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PL CHARACTERISTIC CLASSES AND COBORDISM[1]

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1. **Introduction.** In this note, we announce results on the structure of the unoriented PL cobordism ring, $\mathfrak{N}_*^{\mathrm{PL}}$, and the $\boldsymbol{Z}_2$-characteristic classes for PL bundles, $H^*(B\mathrm{PL})$. All homology and cohomology is with $\boldsymbol{Z}_2$-coefficients, unless otherwise indicated.

There is a sequence of H-space fibrations

$$\Omega(G/\mathrm{PL}) \to \mathrm{PL} \to G \to G/\mathrm{PL} \to B\mathrm{PL} \to BG.$$

The $\boldsymbol{Z}_2$-cohomology of an H-space is a Hopf algebra over the Steenrod algebra. R. J. Milgram [5] has determined $H^*(G)$ and $H^*(BG)$, and D. Sullivan [7] has determined $H^*(G/\mathrm{PL})$ and $H^*(\Omega(G/\mathrm{PL}))$. Our main results, determining the Hopf algebra structure of $H^*(\mathrm{PL})$ and $H^*(B\mathrm{PL})$, follow from spectral sequence arguments, once we have determined the map $H^*(G/\mathrm{PL}) \to H^*(G)$.

W. Browder, A. Liulevicius, and F. P. Peterson [1] have shown that there is an isomorphism of rings $\mathfrak{N}_*^{\mathrm{PL}} \simeq \mathfrak{N}_*^0 \otimes H_*(B\mathrm{PL})//H_*(BO)$, where $\mathfrak{N}_*^0$ is the unoriented, differentiable cobordism ring determined by Thom. Thus our homology computations are sufficient to determine $\mathfrak{N}_*^{\mathrm{PL}}$.

Our methods also determine $H^*(\mathrm{TOP})$ and $H^*(B\mathrm{TOP})$ as Hopf algebras. In fact, these computations are easier than the PL computations. The Kirby-Siebenmann topological transversality theorem implies that $\mathfrak{N}_*^{\mathrm{TOP}} \simeq \pi_*(M\mathrm{TOP}) = \mathfrak{N}_*^0 \otimes H_*(B\mathrm{TOP})//H_*(BO)$ in dimensions $\neq 4$.

2. **Surgery obstructions.** We first recall some results on G/PL (and G/TOP) due essentially to Sullivan [7] (and R. Kirby and L. Siebenmann).

The homotopy groups are given by $\pi_n(G/\mathrm{PL}) = \pi_n(G/\mathrm{TOP}) = P_n$, where $P_n = \boldsymbol{Z}$, 0, $\boldsymbol{Z}_2$, 0 as $n \equiv 0, 1, 2, 3 \pmod 4$, respectively. However, the natural map $G/\mathrm{PL} \to G/\mathrm{TOP}$ has as fibre an Eilenberg-Mac Lane space $K(\boldsymbol{Z}_2, 3)$.

There is a map $G/\mathrm{TOP} \to \prod_{n \geq 1} K(P_n, n) = K(P_*)$ which induces an isomorphism of Hopf algebras over the Steenrod algebra $H^*(K(P_*)) \xrightarrow{\simeq} H^*(G/\mathrm{TOP})$. Let $k_{2n} \in H^{2n}(G/\mathrm{TOP})$ denote the image of the funda-

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mental class $\iota_{2n} \in H^*(K(P_*))$. Then the map $i^*: H^*(G/\mathrm{TOP}) \to H^*(G)$ is completely determined, once the elements $i^*(k_{2n}) \in H^{2n}(G)$ have been computed, $n \geqq 1$.

REMARK 2.1. Denote also by k_{2n} the image of k_{2n} in $H^{2n}(G/\mathrm{PL})$. Since G/PL has one nonzero k-invariant, $\delta\iota_2^2 \in H^5(K(\mathbf{Z}_2, 2), \mathbf{Z})$, where δ is the integral Bockstein, it follows that $k_4 = k_2^2 \in H^4(G/\mathrm{PL})$. However, since $\delta\iota_2^2$ is divisible by 2, $H^*(G/\mathrm{PL})$ and $H^*(G/\mathrm{TOP})$ are abstractly isomorphic as algebras over the Steenrod algebra. Thus $H^*(G/\mathrm{PL})$ has generators $\{k_{2n}, n \neq 2, k_4'\}$ where $k_4' \in H^4(G/\mathrm{PL})$ is a new generator. The Hopf algebra structure of $H^*(G/\mathrm{PL})$ is determined by the coproduct $\Delta(k_4') = k_4' \otimes 1 + k_2 \otimes k_2 + 1 \otimes k_4'$.

Let M^m be a closed manifold, $m \equiv 0 \pmod 2$, and let $\phi: M^m \to G/\mathrm{PL}$ be a map. Then there is a Kervaire surgery obstruction $s_K(M^m, \phi) \in \mathbf{Z}_2$ and a formula for s_K which uniquely characterizes the class $K_{4*-2} = \sum_{n \geqq 1} k_{4n-2}$ [6], [7].

$$2.2 \qquad s_K(M^m, \phi) = \langle V^2(M) \cdot \phi^*(K_{4*-2}), [M] \rangle \in \mathbf{Z}_2$$

where $V^2(M)$ is the square of the total Wu class $V(M) = \sum_{i \geqq 0} V_i(M) \in H^*(M)$.

Let M^m be a $\mathbf{Z}_2$-manifold (that is, $w_1(M)$ is the reduction of an integral class $z_1(M) \in H^1(M, \mathbf{Z})$), $m \equiv 0 \pmod 4$, and let $\phi: M^m \to G/\mathrm{PL}$ be a map. Then there is an index surgery obstruction $s_I(M^m, \phi) \in \mathbf{Z}_2$ and a formula for s_I which uniquely characterizes the class $K_{4*} = \sum_{n \geqq 1} k_{4n}$.

$$2.3 \qquad \begin{aligned} s_I(M^m, \phi) = \langle & V^2(M) \cdot \phi^*(K_{4*}) \\ & + Sq^1((\textstyle\sum V_{2i}(M) Sq^1 V_{2i}(M))\phi^*(K_{4*-2})), [M] \rangle \in \mathbf{Z}_2. \end{aligned}$$

3. **The homology of SG.** A sequence $I = (i_1, \cdots, i_n)$ of positive integers is allowable if $2i_{j+1} \geqq i_j$, all j. We write $kI = (ki_1, \cdots, ki_n)$, $d(I) = \sum_{j=1}^n i_j$, and $e(I) = i_1 - (\sum_{j=2}^n i_j)$. Let $S(n)$ denote the set of allowable sequences I of length n with $e(I) \geqq 0$.

If A is a graded Hopf algebra over $\mathbf{Z}_2$, let A^* denote the dual Hopf algebra, and let $\Lambda(A) \subset A$ denote the Hopf subalgebra generated by squares.

If X is a graded set, introduce Hopf algebras $P(X)$, the polynomial algebra on primitive generators X, $\Gamma(X) = P(X)^*$, the divided power algebra on X, and $E(X)$, the exterior algebra on primitive generators X. Then $E(X) \simeq E(X)^*$. The graded set $s(X)$ will be the set X with elements shifted up one dimension.

The space SG is studied in [4] and [5] by identifying it with the degree one component of $QS^0 = \lim_{n\to\infty} (\Omega^n S^n)$. If $x, y \in H_*(SG)$, denote by $x\cdot y\in H_*(SG)$ their composition product, and denote by $x\underline{*}y\in H_*(SG)$ the loop product $x * y * [-1]$, computed in $H_*(QS^0)$. Here $[q]$ denotes the homology class of a point in the degree q component of QS^0. Let $Q^I=Q^{i_1}\circ\cdots\circ Q^{i_n}$ be the Dyer-Lashof operation. If $I\in S(n)$, let $e_I=Q^I[1]*[1-2^n]\in H_{d(I)}(SG)$. The notation is that of [4]. The following two paragraphs and Lemma 3.1 are reformulations of theorems of [5].

There is an isomorphism of Hopf algebras $H_*(SG)\simeq H_*(SO)\otimes A\otimes(\otimes_{n\geq 2}C_n)$ where $A=\boldsymbol{Z}_2[e_{(i,i)}\mid i\geq 1]$ and $C_n=\boldsymbol{Z}_2[e_I\mid I\in S(n),e(I)\geq 1]$ are Hopf subalgebras of $H_*(SG)$. As an algebra, $H_*(SO)\simeq E(e_i\mid i\geq 1)$. The coproduct is $\Delta(e_n)=\sum_{i+j=n}e_i\otimes e_j$. Further, $e_i=e_*([\boldsymbol{RP}(i)])$, where $e:\boldsymbol{RP}(\infty)\to SO$ is a certain map.

There is an isomorphism of Hopf algebras $H_*(BG)=H_*(BO)\otimes BA\otimes BC_2\otimes(\otimes_{n\geq 3}\overline{B}C_n)$, where $BA=E(s(e_{(i,i)}\mid i\geq 1))$, $BC_n=P(s(e_I\mid I\in S(n),\ e(I)\geq 1))$, and $\overline{B}C_n=P(s(e_I\mid I\in S(n),\ e(I)\geq 2))$.

LEMMA 3.1. *If $x\in H_n(SG)$, then $x=\lambda(x)e_n+\sum y_i'\cdot y_i''+\sum z_j'\underline{*}z_j''$, where $\lambda(x)=0$ or 1 and $y_i', y_i'', z_j', z_j''\in H_*(SG)$ are elements of positive dimensions. In particular, the classes e_i generate $H_*(SG)$ if both products $\cdot$ and $\underline{*}$ are used.*

Next, we need a geometric interpretation of the loop product in $H_*(SG)$. Let x, $y\in H_*(SG)$ be represented by manifolds $\alpha:M^a\to SG$ and $\beta:N^b\to SG$. Then $\alpha*[-1]:M^a\to QS^0$ corresponds to a map $M^a\times S^q\to S^q$ of degree zero on $p\times S^q$, $p\in M$. By transverse regularity, this, in turn, corresponds to a degree zero map $f:M'\to M$ covered by a bundle map $\hat{f}:\nu_{M'}\to\nu_M$. Similarly, let $\beta:N^b\to SG$ correspond to a degree zero map $g:N'\to N$, covered by a bundle map $\hat{g}:\nu_{N'}\to\nu_N$.

LEMMA 3.2. *The element $x\underline{*}y\in H_{a+b}(SG)$ is represented by a map $\alpha\underline{*}\beta:M\times N\to SG$, which corresponds to the degree one normal map*

$$M\times N+M'\times N+M\times N'\xrightarrow{1+(f\times 1)+(1\times g)}M\times N,$$

covered by the bundle map

$$\hat{1}+(\hat{f}\times\hat{1})+(\hat{1}\times\hat{g}),$$

where $+$ indicates disjoint union of manifolds.

4. The map $H^*(G/\mathrm{PL})\to H^*(SG)$.

THEOREM 4.1. *Let $\alpha:M^a\to SG$ and $\beta:N^b\to SG$ be maps, $a+b=2n$. Then*

$$s_K(M \times N, \alpha \stackrel{*}{-} \beta) - s_K(M \times N, \alpha \cdot \beta)$$
$$= \langle (V(M \times N) \cdot \alpha^* \sigma(V) \otimes 1)_n \cdot (V(M \times N) \cdot 1 \otimes \beta^* \sigma(V))_n, [M \times N] \rangle$$
$$= \left\langle V^2(M \times N) \cdot \left(\sum_{r \geq 2} \sum_{i+j=2^r;\, i,j \geq 2} \alpha^* \sigma(w_i) \otimes \beta^* \sigma(w_j) \right), [M \times N] \right\rangle$$
$$\in \mathbf{Z}_2$$

where $\sigma(w_i) \in H^{i-1}(SG)$ *is the suspension of* $w_i \in H^i(BSG)$.

THEOREM 4.2. *Let* $\alpha : M^a \to SG$ *and* $\beta : N^b \to SG$ *be maps,* $a+b=4n$, *where* M^a *and* N^b *are* $\mathbf{Z}_2$*-manifolds. Then*

$$s_I(M \times N, \alpha \stackrel{*}{-} \beta) - s_I(M \times N, \alpha \cdot \beta)$$
$$= \langle Sq^1((V(M \times N) \cdot \alpha^* \sigma(V) \otimes 1)_{2n-1}) \cdot Sq^1((V(M \times N) \cdot 1 \otimes \beta^* \sigma(V))_{2n-1}), [M \times N] \rangle$$
$$= \Bigg\langle V^2(M \times N) \left(\sum_{i \geq 1} \sigma(w_2)^{2^i} \otimes \sigma(w_2)^{2^i} \right) + Sq^1\left(\left(\sum_{i \geq 0} V_{2i}(M) Sq^1 V_{2i}(M) \right) \left(\sum_{r \geq 2} \sum_{i+j=2^r;\, i,j \geq 2} \alpha^* \sigma(w_i) \otimes \beta^* \sigma(w_j) \right) \right), [M \times N] \Bigg\rangle \in \mathbf{Z}_2.$$

Theorem 4.1 is proved using Lemma 3.2, and the result of E. H. Brown, Jr., that the Kervaire surgery obstruction of a degree one normal map may be expressed as a difference of two Arf invariants [2]. To compute this difference in the situation of Lemma 3.2, an additional formula of Brown is needed, which expresses how the Arf invariant of a manifold M^{2n} depends on the choice of a degree one map $S^{q+2n} \to T(\nu_M^q)$. The second equality in Theorem 4.1 is a lengthy computation with Stiefel-Whitney numbers. It is first verified for the products $e_a \stackrel{*}{-} e_b : \mathbf{RP}(a) \times \mathbf{RP}(b) \to SG$, and then the general case is deduced as a corollary.

The proof of Theorem 4.2 is similar to the proof of 4.1, once analogues of Brown's results for the index surgery obstruction for $\mathbf{Z}_2$-manifolds have been established.

As consequences of 2.2, 3.1, and 4.1, and 2.3, 3.1, and 4.2, we have

THEOREM 4.3. *Let* $k_{4n-2} \in H^{4n-2}(G/\mathrm{TOP})$ *be as in* §2. *Then* $i^*(k_{4n-2}) = 0 \in H^*(SG)$ *if and only if* $4n \neq 2^j$. *If* $4n = 2^j$, *then* $\langle i^*(k_{2^j-2}), e_I \rangle = 1$ *if and only if* $I \in S(2)$, $d(I) = 2^j - 2$.

Theorem 4.3 was first proved by Madsen, using the techniques of [3].

THEOREM 4.4. *Let* $k_{4n} \in H^{4n}(G/\mathrm{TOP})$ *be as in* §2. *Then* $i^*(k_{4n}) = 0 \in H^*(SG)$ *if* $4n \neq 2^j$. *If* $4n = 2^j$, *then* $i^*(k_{2^j}) = i^*(k_2^{2^{j-1}})$. *Hence* $i^*(k_{2^j} + k_2{}^{2^{j-1}}) = 0$.

REMARK 4.5. Note that by Remark 2.1 and Theorems 4.3 and 4.4, the map $i^*: H^*(G/\mathrm{PL}) \to H^*(SG)$ is also computed since $\langle i^*(k_4'), e^2_{(1,1)}\rangle = 1$.

Let $K(P_*) = K_1 \times K_2$ where $K_1 = \prod_{n=2^j-2} K(P_n, n)$ and $K_2 = \prod_{n \neq 2^j-2} K(P_n, n)$.

THEOREM 4.6. *There is an exact sequence of Hopf algebras*

$$Z_2 \to H_*(SO) \otimes \Lambda(A \otimes C_2) \otimes \left(\bigotimes_{n \geqq 3} C_n\right) \to H_*(SG)$$
$$\to H_*(G/\mathrm{TOP}) \to \Gamma(W) \otimes H_*(K_2) \to Z_2$$

where W *is a graded set such that there is an isomorphism of Hopf algebras* $H_*(K_1) \simeq \Gamma(W) \otimes \Gamma(I \mid I \in S(2),\ I \neq 2J)$.

5. **The main theorems.** In this section, we state the main results. The proofs consist of (careful) applications of the Eilenberg-Moore or Serre spectral sequence of the fibrations involved.

THEOREM A. *There is an isomorphism of Hopf algebras*

$$H_*(B\mathrm{TOP}) \simeq H_*(BO) \otimes BC_3 \otimes \left(\bigotimes_{n \geqq 4} \overline{B}C_n\right) \otimes E(s(2I \mid I \in S(2)))$$
$$\otimes\ \Gamma(W) \otimes H_*(K_2).$$

Further, $H_*(BO) \otimes BC_3 \otimes (\otimes_{n \geqq 4} \overline{B}C_n) \simeq \text{image } (H_*(B\mathrm{TOP}) \to H_*(BG))$, *and* $\Gamma(W) \otimes H_*(K_2) \simeq \text{image } (H_*(G/\mathrm{TOP}) \to H_*(B\mathrm{TOP}))$.

THEOREM B. *There is an isomorphism of Hopf algebras*

$$H_*(S\mathrm{TOP}) \simeq H_*(SO) \otimes \Lambda(A \otimes C_2) \otimes \left(\bigotimes_{n \geqq 3} C_n\right) \otimes \Gamma(V) \otimes H_*(\Omega K_2)$$

where V *is a graded set such that, as algebras,* $\Gamma(V) = E(s^{-1}(W))$.

The computation of $H_*(B\mathrm{PL})$ is more complicated because of Remarks 2.1 and 4.5. First, we need more notation. Let

$$Y = \{2^i(2,1,1) \mid i \geqq 0\} \cup \{2^i(2^{j+1}+1, 2^j+1, 2^j) \mid i,j \geqq 0\}$$
$$\cup \{2^i(2^{j+k+1} + 2^j + 1, 2^{j+k} + 2^j, 2^{j+k}) \mid i,j,k \geqq 0\} \subset S(3).$$

Let $X=S(3)-Y$, and let $X_0=\{I\in X \mid e(I)=0\}$. Let $X_1=X-X_0$. Finally, let $K_2' = \prod_{n\neq 4,2^j-2} K(P_n, n)$.

THEOREM C. *There is an isomorphism of Hopf algebras*

$$H_*(B\mathrm{PL}) \simeq H_*(BO) \otimes P(\boldsymbol{Z}) \otimes \Big(\bigotimes_{n\geq 5} \overline{B}C_n \Big) \otimes P(s(X_1)) \otimes \Gamma(W)$$
$$\otimes\, H_*(K_2') \otimes E(s(X_0)) \otimes E(s(2I \mid I \in Y)).$$

where $\boldsymbol{Z}$ *is a graded set such that* $P(\boldsymbol{Z}) \otimes \Lambda(P(s(X_1))) \simeq BC_4$.

THEOREM D. *There is an isomorphism of Hopf algebras*

$$H_*(S\mathrm{PL}) \simeq H_*(SO) \otimes \Big(\bigotimes_{n\geq 4} C_n \Big) \otimes \boldsymbol{Z}_2[e_I \mid I \in X]$$
$$\otimes\, \Lambda(\boldsymbol{Z}_2[e_I \mid I \in Y]) \otimes \Gamma(V) \otimes H_*(\Omega K_2').$$

REMARK. It is easy to read off the dual Hopf algebras $H^*(B\mathrm{TOP})$ and $H^*(B\mathrm{PL})$ and the cobordism ring $\mathfrak{N}_*^{\mathrm{PL}} \simeq \mathfrak{N}_*^0 \otimes (H_*(B\mathrm{PL})//H_*(BO))$ from Theorems A and C.

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