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AN EXISTENCE THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES[1]

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ABSTRACT. We consider nonlinear ordinary differential equations in Banach spaces. A local existence theorem for the Cauchy problem is given when the equation is continuous in the weak topology. The theorem can be extended to set-valued differential equations in Banach spaces.

Let B be a Banach space and let $F:(0, 1)\times B\rightarrow B$. If B is finite dimensional and F is continuous in a neighborhood of $(t_0, x_0)\in(0, 1)\times B$, then by the Peano existence theorem there exists a function $\phi(t)$ defined on a subinterval of $(0, 1)$ such that

$$\phi'(t) = F(t, \phi(t)) \quad \text{and} \quad \phi(t_0) = x_0.$$

Dieudonné [1] and Yorke [2] have shown, by means of examples, that continuity alone, of the function F, is not sufficient to prove a local existence theorem in the case where B is infinite dimensional. Other authors, for example [3] and [4], have extended the Peano theorem to infinite-dimensional spaces but with additional assumptions. We have found that by replacing strong continuity with weak continuity and assuming the range of F to be bounded we may obtain an existence theorem.

Let B be a separable reflexive Banach space with norm $\|\cdot\|$ and let B^* be its dual space. Let B_w denote the space B with the weak topology and let $\{f_i\}$ be a countable dense subset in B^*. By Δ we mean a subinterval of $T=(0, 1)$.

DEFINITION 1. A function $F: T\times B\rightarrow B$ is said to satisfy condition (I) if, at each $(t_0, x_0)\in T\times B$,

$$F(t_0, x_0) = \bigcap_{N=1}^{\infty} \operatorname{cl} \operatorname{co} F_N(t_0, x_0)$$

where

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$$F_N(t_0, x_0) = \bigcup \left\{F(t, x): |t - t_0| < \frac{1}{N}, \right.$$
$$\left. |f_i(x - x_0)| < \frac{1}{N}, i = 1, \cdots, N\right\}$$

and cl co $F_N(t_0, x_0)$ denotes the closed convex hull of $F_N(t_0, x_0)$.

If B is finite dimensional, condition (I) is equivalent to saying that F is continuous. If $F: T \times B_w \to B_w$ is continuous, then F satisfies condition (I). If F takes on set values in B and B is finite dimensional, then condition (I) is equivalent to saying that F is upper semicontinuous in the sense of Cesari [5], [6]. Thus, condition (I) is a reasonable generalization of continuity and it will also apply to set-valued differential equations.

We consider the differential equation

$$\text{(E)} \qquad \dot{x} = F(t, x), \quad \text{where} \quad F: T \times B \to B.$$

DEFINITION 2. A solution of (E) on Δ is a function $\phi(t)$ defined on Δ such that $\phi(t)$ is weakly continuous (i.e., for every $f \in B^*$, $f(\phi(t))$ is a continuous real-valued function), and for almost every $t \in \Delta$,

$$D\phi(t) = F(t, \phi(t))$$

where $D\phi(t)$ is the weak limit of $(\phi(t+h) - \phi(t))/h$ as $h \to 0$.

THEOREM 1. *Let $F: T \times B \to B$ satisfy condition* (I) *and let $\phi: \Delta \to B$ be weakly continuous. Then $\phi(t)$ is a solution of* (E) *on Δ if and only if, for every $N \geqq 1$, $t \in \Delta$, there exists $\eta > 0$ such that*

$$0 < h < \eta \Rightarrow \frac{\phi(t + h) - \phi(t)}{h} \in \text{cl co } F_N(t, \phi(t)).$$

THEOREM 2. *Let $F: T \times B \to B$ and let $(t_0, x_0) \in T \times B$. Assume that in a neighborhood of (t_0, x_0), F satisfies condition* (I) *and is bounded in norm. Then there exists a solution $\phi(t)$ of* (E) *on some interval Δ such that $\phi(t_0) = x_0$. Further, $\phi(t)$ is absolutely continuous and $\phi'(t) = F(t), \phi(t))$ a.e. on D (where $\phi'(t)$ is the strong limit of $(\phi(t+h) - \phi(t))/h$ as $h \to 0$).*

The method of polygonal approximations is used in this proof. The approximations converge weakly to a weakly continuous limit function. By Theorem 1, we are able to show that this function is a solution of (E) and that it is Lipschitz. Then by a result of Pettis [7] it follows that ϕ has a strong derivative a.e.

COROLLARY. *Let* $F: T\times B_w \to B_w$ *and let* $(t_0, x_0) \in T\times B_w$. *Assume that in a neighborhood of* (t_0, x_0), F *is continuous and bounded in norm. Then there exists a solution* $\phi(t)$ *of* (E) *defined on some interval* Δ *such that* $\phi(t_0) = x_0$, $\phi(t)$ *is absolutely continuous on* Δ, *and* $\phi'(t) = F(t, \phi(t))$ *a.e. on* Δ.

We have also proved Theorem 2 in the case where F takes values in the closed convex subsets of B (i.e. for set-valued differential equations) and we have proved theorems on continuous dependence on initial conditions and closure of families of solutions.

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