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C^1 PARTITIONS OF UNITY ON NONSEPARABLE HILBERT SPACE

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The main result is that every Hilbert space admits continuously differentiable partitions of unity. We sketch a proof of the key proposition. Details will appear in [4].

Much more is known for separable Banach spaces. R. Bonic and J. Frampton [1] showed that if there are any nontrivial C^k (i.e., k continuous Fréchet derivatives) on E, a separable Banach space, then E admits C^k partitions of unity. Thus separable Hilbert space, l^n, for n an even integer, and c_0 admit C^∞ partitions of unity. C^∞ partitions of unity on separable l^2 were first constructed by James Eells; a proof appears in [2].

Let R^n be n-dimensional Euclidean space. Let

$$C^k_{1,M} = \left\{ f \;\middle|\; f \in C^k(R^n, R), \sup_{x \neq y} \left(\|D^k f(x) - D^k f(y)\| / \|x - y\| \right) \leqq M \right\}.$$

If A is a closed subset of R^n, call f a $C^k_{1,M}$ A-selecting function if $f \in C^k_{1,M}$, $0 \leqq f(x) \leqq 1$, $C^k_{1,M}(x) = 1$ if $x \in A$ and $f(x) = 0$ if $d(x, A) \geqq 1$. By smoothing out $\sup(0, 1 - d(x, A))$ we can always find a $C^k_{1,M}$ A-selecting function provided M is large enough. For $k = 0$, $f(x) = \sup(0, 1 - d(x, A))$ has smallest M namely 1. For $k = 1$ and 2, we have the following:

THEOREM 1. *Let $A = \{x \mid x_i \leqq 0, \|x\| \leqq 1, i = 1, \cdots, n\}$. Then if f is a $C^2_{1,M}$ A-selecting function, $n > M^2 + 36M^4$.*

COROLLARY 1. *The Whitney Extension Theorem fails for separable Hilbert space.*

THEOREM 2. *If A is a closed subset of H, any Hilbert space, then there exists a $C^1_{1,4}$ A-selecting function, f, and if $g \in C^1_{1,4}(H, R)$, $g(x) = 1$ for x in A and $0 \leqq g(x) \leqq 1$, then $f(x) \leqq g(x)$.*

The key to the proof of Theorem 2 is

PROPOSITION 1. *Theorem 2 is true if H is finite dimensional and $A = F$ a finite subset.*

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PROOF. If $F=\{a\}$, a single point, then $f(x)=n(\|x-a\|)$ where $n(t)=1-2t^2$, for $0\leqq t\leqq \frac{1}{2}$, $n(t)=2(1-t)^2$, for $\frac{1}{2}\leqq t\leqq 1$, and $n(t)=0$, for $1\leqq t$.

In general if $S\subset F$ let $S_*=\{y\mid \|y-p\|=\|y-q\|(1, \|y-z\|)$ for all p, $q\in S$ and $z\in F$, $z\notin S\}$. Let $K=\{S\mid S_*\neq\emptyset\}$, and, for $S\in K$, let r_S be the distance from the vertices of S to the point equidistant from the points of S in the plane determined by S. Then S_* is closed convex and, for $S\in K$, the planes determined by S and S_* are perpendicular. Let $D_S(x)$ and $D_{S_*}(x)$ be the distance from x to these planes. We obtain a $C^1_{1,4}$ F-selecting function on a mixture of the complex K and $\bigcup_{S\in K}S_*$. Let $G=\{x\mid d(x, F)=1\}$ and for $S\in K$ define:

$$T_S=\{x\mid x=(y+z)/2 \quad \text{for some } y\in S,\ z\in S_*\},$$
$$Q_S=\{x\mid x=ty+(1-t)z \quad \text{for some } y\in S,\ z\in S_*\cap G,\ 0\leqq t\leqq \tfrac{1}{2}\}.$$

Then it can be shown that the T_S's and Q_S's are closed, have interiors, have $(n-1)$-dimensional intersections and that $\bigcup_{S\in K}Q_S\cup T_S=\{x\mid d(x, F)\leqq 1\}$.

We define

$$f_{T_S}(x) = 1 - r_S^2 - 2D_S^2(x) + 2D_{S_*}^2(x),$$
$$f_{Q_S}(x) = 2D_{S_*}^2(x) + 2(D_S(x) - r_S)^2.$$

It is easy to show that f_{T_S} and f_{Q_S} are C^2, that $\|D^2f_{T_S}(x)\|=\|D^2f_{Q_S}(x)\|=4$ and that $f_{T\{p\}}=1$ for $p\in F$. It is also possible to show that f_{T_S}, f_{Q_S} and Df_{T_S}, Df_{Q_S} agree wherever their domains intersect and that $f_{Q_S}(x)=Df_{Q_S}(x)=0$ if $x\in G$. Hence the function $f(x)=f_{T_S}(x)$ if $x\in T_S$, $f(x)=f_{Q_S}(x)$ if $x\in Q_S$, for $S\in K$, and $f(x)=0$ if $d(x, F)\geqq 1$ is $C^1_{1,4}$ F-selecting. The second part of the proposition can be established by first proving $f(x)\leqq g(x)$, for $x\in T_{\{p\}}$, $p\in F$, and then showing that $f\leqq g$ on T_S for dim $S<k$ implies $f<g$ on T_S for dim $S=k$. $f\leqq g$ on T_{Q_S} follows from this. The figure illustrates the partitions when F is three points in R^2, $F_*\neq\emptyset$ and $F_*\in\text{Cohull}(F)$.

We now prove Theorem 2. We need the following lemma:

LEMMA 1. *If* $\text{Lim}_{p\in D}f_p(x)=f(x)$ *for all* x *in some Banach space* E *where* $f_p\in C^1_{1,4}(E, R)$, *then* $f\in C^1_{1,4}(E, R)$.

PROOF OF THEOREM 2. If A is a closed subset of H, direct pairs (F, M) where F is a finite subset of A and M is a finite-dimensional plane in H containing F by $(F, M)\leqq(F', M')$ if $F\subset F'$ and $M\subset M'$. Then find $f_{F,M}$, $C^1_{1,4}$ F-selecting on M. By the second part of the

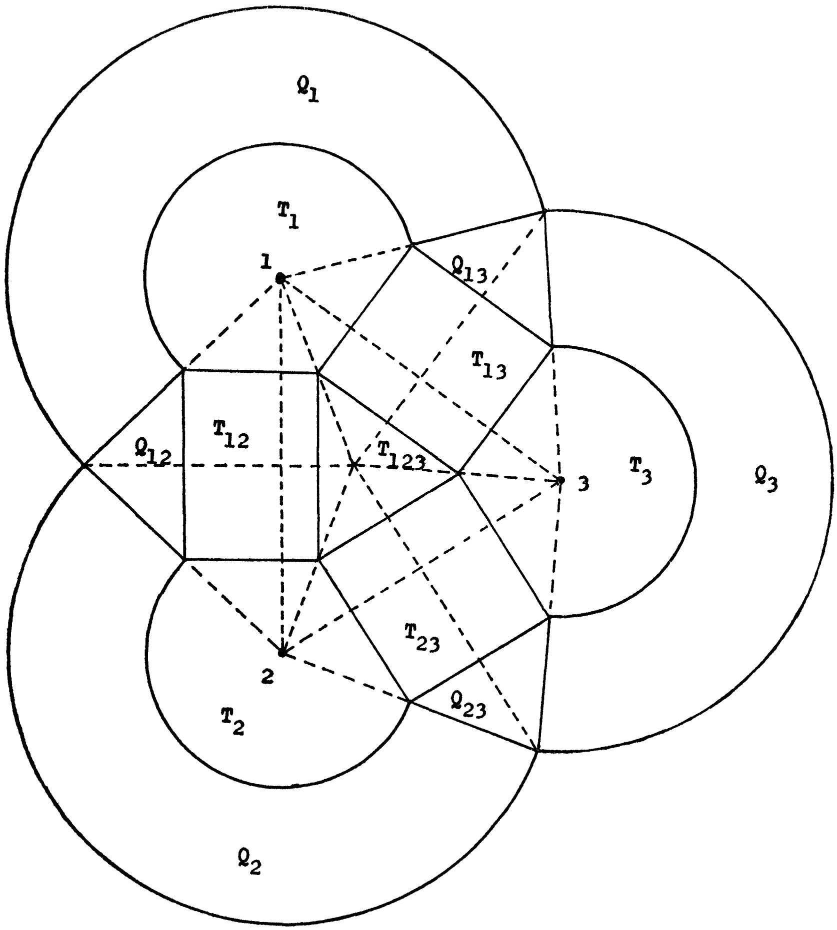

FIGURE 1

proposition, $(F, M) \leqq (F', M')$ implies $f_{F,M} \leqq f_{F',M'}|_M$. Hence the net is monotone and the limit exists since $f_{F,M} \leqq 1$. By the lemma, $\mathrm{Lim}_{F,M} f_{F,M}|_{M'}$ is $C^1_{1,4}$ for all M', hence $\mathrm{Lim}_{F,M} f_{F,M} = f(x)$ is $C^1_{1,4}$. That f is A-selecting and that the second part of the theorem holds is obvious.

COROLLARY 2. *If U is open in H a Hilbert space, there exists a $C^1_{1,\cdot}$ (H, R) function with* $0 \leqq f(x)$ *and* $U = \{x \mid f(x) > 0\}$.

PROOF. Apply Theorem 2 to the sets $A_n = \{x \mid d(x, \text{complement of } U) \leqq 2^{-n}\}$ etc.

COROLLARY 3. *Any Hilbert space admits C^1 partitions of unity.*

COROLLARY 4. *$C^1(H, F)$ is dense in $C^0(H, F)$ for any Hilbert space H and any Banach space F.*

REMARKS. If A is convex then $n(d(x, A))$ is the $C^1_{1,4}$ A-selecting function. If the Euclidean norm on R^n is replaced by the c_0 norm, then given any $M>0$ there is no $C^1_{1,M}$ $\{0\}$-selecting function in n-dimensional space for $n>2^M$. This follows from a result of the author contained in [3].

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