GENERALIZED SOLUTIONS OF QUASILINEAR, DIFFERENTIAL INEQUALITIES. I. ELLIPTIC OPERATORS

BY NEIL S. TRUDINGER

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Introduction. The purpose of this note, together with its sequel on parabolic inequalities, is to present various results describing the local and global behaviour of generalized solutions, subsolutions and super-solutions associated with quasilinear, second order differential operators of the forms

$$(1) \qquad Qu = \operatorname{div} \mathfrak{A}(x, u, Du) + \mathfrak{B}(x, u, Du),$$

$$(2) \qquad Pu = \operatorname{div} \mathfrak{A}(x, t, u, Du) + \mathfrak{B}(x, t, u, Du) - D_t u.$$

Here $x = (x_1, \cdots, x_n)$ represents a point in a Euclidean n-space E^n; t denotes a time variable; Du is the spatial gradient of the strongly differentiable function u; $\mathfrak{A}$ and $\mathfrak{B}$ are respectively measurable, n-vector and scalar functions of their arguments and div denotes the spatial divergence.

The results announced below are a selective sampling from the work [14] and significantly extend the theory of operators Qu, Pu as previously developed by such authors as De Giorgi [2], Moser [5], [6], Ladyženskaya and Ural'ceva [3], [4], Serrin [1], [7], Stampacchia [8], [9], Aronson [1], Trudinger [10], [11] and others. The present note discusses the operators Qu, although we remark that the theory for Pu is largely derived through extrapolation of the Qu methods. The *local* results for Qu are in general specializations of their extensions to Pu. The proofs of the theorems here, as supplied in [14], involve some interesting new test function techniques.

Let us restrict ourselves here to a coefficient structure determined by polynomials in $|u|$ and $|Du|$ with coefficients in L_p spaces. For a more general type of structure see [13]. Let Ω be a bounded domain in E^n and $G \subset \Omega \times E^{n+1}$. We define the following structural inequalities, A_1, A_2, B, to be satisfied for all $x, u, p \in G$:

$$A_1: \quad |\mathfrak{A}(x, u, p)| \leqq \lambda(x)a(x)|p|^{\alpha-1} + |c(x)u|^{\alpha-1} + |f(x)|^{\alpha-1},$$

$$(3) \quad A_2: \quad p \cdot \mathfrak{A}(x, u, p) \geqq |\lambda(x)p|^{\alpha} - |e(x)u|^{\alpha} - |h(x)|^{\alpha},$$

$$B: \quad |\mathfrak{B}(x, u, p)| \leqq b_0|\lambda(x)p|^{\alpha} + \lambda(x)b(x)|p|^{\alpha-1}$$
$$+ |d(x)|^{\alpha}|u|^{\alpha-1} + |g(x)|^{\alpha},$$

where $\alpha \geqq 1$, b_0 is a nonnegative constant and the coefficients a, $b, \cdots, h$, together with λ^{-1} are nonnegative functions lying in various L_p spaces. For our purposes below, we may specify here: $\lambda^{-1} \in L_t(\Omega)$, $a, b, c, d, e, g \in L_s(\Omega), f, h \in L_r(\Omega)$ where r, s, t satisfy

$$(4) \qquad \frac{1}{t} + \frac{1}{s} < \frac{1}{n} \leqq \frac{1}{t} + \frac{1}{\alpha} \leqq 1, \qquad \frac{1}{t} + \frac{1}{r} = \frac{1}{q} \leqq \frac{1}{\alpha} \cdot$$

The results in [14] are by no means confined by condition (4). For instance, in many cases the strict inequality in (4) may be weakened and the lower bound on n implied there is not necessary.

In terms of the structure, we may define a convenient notion of solution (for a more natural and general notion see [14]). Let $V^0(\Omega)$ denote the completion of $C_0^1(\Omega)$ under the $L_\alpha(\Omega)$ norm of $|Du|$ with respect to the measure $d\mu = \lambda^\alpha dx$ and let

$$V^{\mathrm{loc}}(\Omega) = \{u; \phi u \in V^0(\Omega) \text{ for all } \phi \in C_0^1(\Omega)\}.$$

Set

$$(5) \qquad Q(u, \phi) = \int_\Omega \{D\phi \cdot \mathfrak{A}(x, u, Du) - \phi \mathfrak{B}(x, u, Du)\} \, dx.$$

A function, $u \in V^{\mathrm{loc}}(\Omega)$, satisfies $Qu \geqq 0$ $(Qu \leqq 0)$ in Ω if $-\infty < Q(u, \phi)$ $\leqq 0$ $(\infty > Q(u, \phi) \geqq 0)$ for all $\phi \geqq 0$, $\in V^0(\Omega)$. Furthermore, u is said to satisfy $u \leqq L$ on $\partial\Omega$, in the sense of $V^0(\Omega)$, if the function $v = \sup(u - L, 0) \in V^0(\Omega)$.

Global estimates. Assume now, that conditions A_2 and B hold on $G = \Omega \times E^{n+1}$ with $b_0 = 0$. Then we have

THEOREM 1. *Let $u \in V^{\mathrm{loc}}(\Omega)$ satisfy $Qu \geqq 0$ in Ω and $u \leqq L < \infty$ on $\partial\Omega$. Then setting $v = \sup(u - L, 0)$ we have*
    (i) *if $q > n$, v is in $L_\infty(\Omega)$;*
    (ii) *if $q = n$, v is in the Orlicz space $L_\phi(\Omega)$ where $\phi = e^{|t|} - 1$;*
    (iii) *if $q < n$, v is in $L_p(\Omega)$ where $1/p = 1/q - 1/n$.*
    In each case the norm of v may be estimated in terms of L, n, r, s, t, α, $\|u\|_{L_\alpha(\Omega)}, |\Omega|$ and the relevant norms of $\lambda^{-1}, a, \cdots, h$.

To obtain an *a priori* estimate for sup u, independent of $\|u\|_{L_\alpha(\Omega)}$, we assume, additionally to case (i) above, that $d = e = 0$.

THEOREM 2. *Let* $u \in V^{\mathrm{loc}}(\Omega)$ *satisfy* $Qu \geqq 0$ *in* Ω. *Then*

(6)
$$\sup_{\Omega} u \leqq \sup_{\partial\Omega} u + C\{\|g\|^{\alpha/(\alpha-1)} + \|h\|\}$$

where the constant C *depends on* n, r, s, t, $|\Omega|$, $\|\lambda^{-1}\|$, $\|b\|$ *and* $g = 0$ *if* $\alpha = 1$.

Corresponding results for supersolutions and consequently solutions follow automatically. Note also that if $|\mathfrak{B}(x, u, p)|$ is replaced by sign $u \cdot \mathfrak{B}(x, u, p)$ in condition B, then Theorem 2 holds for the function $u^+ = \sup(u, 0)$.

Local estimates. A local version of Theorem 1 is true if A_1, A_2 and B all hold on $G = \Omega \times E^{n+1}$, with $b_0 = 0$. See [14] for the detailed statement. Let us instead consider bounded solutions and assume that conditions A_1, A_2 and B hold on sets $G_M = \Omega \times (-M, M) \times E^n$ with $r = s$, $\alpha > 1$. For supersolutions the following estimate, which reduces to a weak Harnack inequality when $f = g = h = 0$, is derived.

THEOREM 3. *Let* u *be a bounded, nonnegative function in* $V^{\mathrm{loc}}(\Omega)$ *satisfying* $Qu \leqq 0$ *in* Ω *and let* B *be a ball in* Ω. *Then for any smaller concentric balls,* B_1 *and* B_2, *we have*

(7)
$$\left(|B_1|^{-1} \int_{B_1} u^{\alpha-1}\right)^{1/(\alpha-1)} \leqq C\left(\inf_{B_2} u + k\right)$$

where the constant C *depends on* n, s, t, $b_0 \sup_B u$, $|B|$, $|B_1|$, $|B_2|$ *and the norms of* λ^{-1}, a, b, $\cdots$, e *over* B *while* $k = |B|^{1/n-1/q}(\|f\|_B + \|h\|_B) + (|B|^{1/n-1/q}\|g\|_B)^{\alpha/(\alpha-1)}$.

Some important corollaries of Theorem 3 are:

I. *a strong maximum principle* for subsolutions of Q when a homogeneous structure prevails,

II. *a Harnack inequality* for locally bounded solutions,

III. *a Hölder estimate* for locally bounded solutions when an additional coefficient restriction, namely that

$$\Lambda(B) = |B|^{-(1/t+1/s)}\|\lambda^{-1}\|_B\|a\|_B$$

is bounded uniformly on balls $B \subset \Omega$, is satisfied,

IV. *the semicontinuity* of subsolutions and supersolutions, also under the above restriction. This result also extends to the case $\alpha = 1$ and consequently embraces the minimal surface equation.

Finally, we note that the theorems above are new even in the nondegenerate case, $t = \infty$, and also for linear elliptic operators of the form

$$(8) \qquad Lu = \frac{\partial}{\partial x_i} (a^{ij}(x)u_{x_j} + a^i(x)u + f^i) + b^i(x)u_{x_j} + a(x)u + f,$$

ellipticity meaning that the matrix $[a^{ij}(x)]$ is positive almost everywhere in Ω. New existence and uniqueness theorems for Lu follow from Theorem 2.

References

1. D. G. Aronson and J. Serrin, *Local behavior of solutions of quasilinear parabolic equations*, Arch. Rational Mech. Anal. **25** (1967), 81–122. MR **39** #5952.

2. E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43. MR **20** #172.

3. O. A. Ladyženskaja and N. N. Ural'ceva, *Linear and quasilinear equations of elliptic type*, "Nauka", Moscow, 1964; English transl., Academic Press, New York, 1968. MR **35** #1955; MR **39** #5941.

4. O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasilinear parabolic equations*, "Nauka", Moscow, 1967; English transl., Transl. Math. Monographs, vol. 23, Amer. Math. Soc., Providence, R. I., 1968. MR **39** #3159a, b.

5. J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591. MR **28** #2356.

6. ——, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134. MR **28** #2357.

7. J. Serrin, *Local behaviour of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302. MR **30** #337.

8. G. Stampacchia, *Contributi alla regolarizzazione delle soluzioni dei problemi al contorno per equazione del secondo ordine ellittiche*, Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 223–245. MR **23** #A2616.

9. ——, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), fasc. 1, 189–258. MR **33** #404.

10. N. S. Trudinger, *On Harnack type inequalities and their application to quasi-linear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747. MR **37** #1788.

11. ——, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math. **21** (1968), 205–226. MR **37** #1758.

12. ——, *On the regularity of generalized solutions of linear, nonuniformly elliptic equations*, Arch. Rational Mech. Anal. **41** (1971) (to appear).

13. ——, *Continuity of weak solutions of quasilinear equations*, Indiana Univ. Math. J. (to appear).

14. ——, *Local and global behaviour of generalized solutions of quasilinear, differential inequalities* (in preparation).

University of Minnesota, Minneapolis, Minnesota 55455

University of Queensland, St. Lucia, Queensland 4067, Australia