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JORDAN TRIPLE SYSTEMS, R-SPACES, AND BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. In this note, we establish a one-to-one correspondence between compact Jordan triple systems (see below for the definition) and symmetric R-spaces (i.e., symmetric spaces which are quotients of semisimple Lie groups by parabolic subgroups, see [7]). We obtain a simple geometric characterization of symmetric R-spaces among compact symmetric spaces. The noncompact dual of a symmetric R-space may be realized as a bounded domain D in a real vector space. There is a one-to-one correspondence between boundary components of D and idempotents of the corresponding Jordan triple system. Using this, we generalize the results of Wolf-Koranyi [8] to the real case. In particular, the image of D under a generalized Cayley transformation is the real equivalent of a Siegel domain of type III.

1. Jordan triple systems.

A Jordan triple system (=JTS) (see [2], [3]) is a vector space V together with a trilinear map $V\times V\times V \to V$, $(x, y, z)\mapsto \{xyz\}$, satisfying the following identities:

$$\{xyz\} = \{zyx\}, \tag{1}$$

$$\{uv\{xyz\}\} = \{\{uvx\}yz\} - \{x\{vuy\}z\} + \{xy\{uvz\}\}. \tag{2}$$

For $x, y\in V$ we define the linear map $L(x, y)$ of V by $L(x, y)(z) = \{xyz\}$. A finite-dimensional real JTS is called *compact* if the quadratic form $x\mapsto \operatorname{trace} L(x, x)$ is positive definite. From now on, V will denote a compact JTS. Then V becomes a Euclidean vector space with the scalar product $(x, y) = \operatorname{trace} L(x, y)$. By (2), the vector space $\mathfrak{H}$ spanned by $\{L(x, y): x, y\in V\}$ is a Lie algebra of linear transformations of V, closed under taking transposes with respect to $(\, , \,)$. Thus the contragredient $\mathfrak{H}$-module V' of V may be identified with V as a vector space, and $X\cdot v' = -{}^tX(v')$, for $X\in\mathfrak{H}$ and $v'\in V'$.

THEOREM 1 (KOECHER). (a) *$\mathfrak{L} = V+\mathfrak{H}+V'$ becomes a semisimple Lie algebra with the definitions*

$$[X, Y] = XY - YX, \qquad [X, v] = -[v, X] = X\cdot v,$$

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for X, Y *in* $\mathfrak{H}$ *and* $v \in V \cup V'$;

$$[V, V] = [V', V'] = 0, \qquad [u, v'] = -2L(u, v),$$

for $u \in V$ *and* $v' \in V'$.

(b) $Z = -\mathrm{Id}_V$ *belongs to* $\mathfrak{H}$, $(\mathrm{ad}\ Z)^3 = \mathrm{ad}\ Z$, *and the* -1-, 0-, $+1$-*eigenspaces of* ad Z *are* V, $\mathfrak{H}$, V'.

It is easily seen that the map $\tau: X \mapsto -{}^tX$ $(X \in \mathfrak{H})$, $v \mapsto v'$, $v' \mapsto v$, is a Cartan involution of $\mathfrak{L}$, and that $\sigma | \mathfrak{H} = +1$, $\sigma | V + V' = -1$ defines an involutive automorphism σ of $\mathfrak{L}$ commuting with τ.

2. **Symmetric R-spaces.** Keeping the above notations, let L be the centerfree connected Lie group with Lie algebra $\mathfrak{L}$, let H be the centralizer of Z in L, let U be the maximal compact subgroup of L determined by τ, and let $K = U \cap H$. Then K lies between the full fixed point set of σ in U and its identity component. The normalizer P of V in L is parabolic, and we have $U/K \cong L/P$. Thus $M = U/K$ is a symmetric R-space (cf. [7]).

THEOREM 2. *The map* $V \mapsto M$ *establishes a one-to-one correspondence between isomorphism classes of compact JTS and symmetric R-spaces.*

Let M be a compact symmetric space, o a point of M, and A a maximal torus of M containing o. The tangent space of A at o is denoted by $T_o(A)$. Then $\Lambda(M) = \{v \in T_o(A) : \mathrm{Exp}\ v = 0\}$ is a lattice in $T_o(A)$, the *unit lattice* of M (with respect to A). We say that M has *cubic* unit lattice if there exists a Riemannian metric on M, invariant under all symmetries, and an orthonormal basis $e_1, \cdots, e_r$ of $T_o(A)$ with respect to this metric, such that $\Lambda(M) = \sum \mathbf{Z} \cdot e_i$.

THEOREM 3. *A compact symmetric space is a symmetric R-space if and only if it has cubic unit lattice.*

As a special case, we obtain the following characterization of the classical groups: *A compact connected Lie group is a direct product of the groups* $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$ *if and only if it has cubic unit lattice.* The fact that symmetric R-spaces have cubic unit lattice is contained in [7]. Theorem 3 may be used to give a global classification of symmetric R-spaces (and hence of compact JTS). The classification has been obtained by different methods in [1] and [4].

3. **Bounded symmetric domains.** Keeping the previous notations, let G be the connected fixed point set of $\sigma\tau$ in L. Then $M^* = G/K_0$ is the noncompact dual of M. There is an imbedding $\zeta: M^* \to V$ such that $g \equiv \exp(\zeta(gK_0))$ mod P, for $g \in G$. The image $D = \zeta(M^*)$ is a

bounded domain in V (see [4], [7]) which inherits a Riemannian metric from M^*. This generalizes the Harish-Chandra imbedding of a Hermitian symmetric space of noncompact type. Following Pjateckii-Shapiro [5], two points x, y in $\bar{D}$ (the closure of D in V) will be called equivalent if there exist sequences (x_n), (y_n) in D, converging to x, y respectively, such that the Riemannian distance of x_n and y_n remains bounded. The equivalence classes are the *metric boundary components* of D. We also define *affine boundary components* as follows. By a segment in V we mean a set of the form $\{a+tb:0<t<1\}$ where $a, b\in V$. Then a subset F of $\bar{D}$ is called an affine boundary component if (1) every segment contained in $\bar{D}$ and meeting F is contained in F, (2) no proper nonempty subset of F satisfies (1).

An element c of V is called an *idempotent* if $\{ccc\}=c$. Part of the following theorem is due to K. Meyberg (unpublished).

THEOREM 4. (a) *There is a Peirce-decomposition* $V=V_1(c)+V_{1/2}(c)+V_0(c)$ *where* $V_i(c)$ *is the eigenspace of* $L(c, c)$ *corresponding to the eigenvalue* i.

(b) *With* $x\circ y=\{xcy\}$, $V_1(c)$ *is a real semisimple Jordan algebra with unit element* c. *The map* $x\mapsto \bar{x}=\{cxc\}$ *is a Cartan involution of* $V_1(c)$; *in particular*, $V_1^+(c)=\{x\in V_1(c):\bar{x}=x\}$ *is a formally real Jordan algebra.*

(c) *With the induced multiplication*, $V_0(c)$ *is a compact JTS.*

Now we can describe the relation between boundary components and idempotents.

THEOREM 5. *Metric and affine boundary components coincide; they are precisely the sets* $F_c=c+(D\cap V_0(c))$ *where* c *is an idempotent.* $D_c=D\cap V_0(c)$ *is the bounded symmetric domain belonging to the compact JTS* $V_0(c)$.

This allows us to recover all the results of [8]. In particular, the space of boundary components of a given type is a fibre bundle over a compact symmetric space, and the stability group of a boundary component in G is parabolic. Finally, we define the Cayley transformation belonging to an idempotent c (resp. a boundary component F_c) to be $\gamma_c=\exp \frac{1}{4}\pi(c+\tau(c))$. For $z\in D_c$ and $x, y\in V_{1/2}(c)$ set $\Phi_z(x, y)=\{x, (\mathrm{Id}+\mu(z))^{-1}(y), c\}$ where $\mu(z)(y)=2\{cyz\}$. Also for $u\in V_1(c)$ let $\mathrm{Re}\ u=\frac{1}{2}(u+\bar{u})\in V_1^+(c)$, and denote by Y the interior of the set of squares of $V_1^+(c)$. This is the self-dual cone (domain of positivity) associated with the formally real Jordan algebra $V_1^+(c)$.

THEOREM 6. *The image of D under the Cayley transformation γ_c is the set of all $x+y+z \in V_1(c)+V_{1/2}(c)+V_0(c)$ such that $z \in D_c$ and* $\operatorname{Re}(x-\Phi_z(y,y)) \in Y$, *a real Siegel domain of type* III.

In the special case where c is a maximal idempotent, i.e., $V_0(c)=0$, we have $\gamma_c(D)=\{x+y \in V_1(c)+V_{1/2}(c) : \operatorname{Re} x-\{yyc\} \in Y\}$, a real Siegel domain of type II. This result is due to Takeuchi [7], see also [6].

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