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TRANSLATION-INVARIANT LINEAR FORMS AND A FORMULA FOR THE DIRAC MEASURE

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Following Schwartz [2] we denote by $\mathfrak{D}$, $\mathcal{E}$ and $\mathcal{S}$ the complex vector spaces of all complex-valued infinitely differentiable functions ϕ on $\boldsymbol{R}^n$ where the functions of $\mathfrak{D}$ have compact supports, the functions of $\mathcal{E}$ have arbitrary supports, and the functions of $\mathcal{S}$ (along with all their derivatives) are rapidly decreasing at infinity. We equip each of these spaces with its usual locally convex topology. These spaces and their duals $\mathfrak{D}'$, $\mathcal{E}'$ and $\mathcal{S}'$ are translation-invariant in the sense that the translated function (or distribution) $\phi_h(t) \equiv \phi(t-h)$ belongs to the space whenever ϕ does. We say that a (not necessarily continuous) linear form L on any of these spaces is "translation-invariant" if $L(\phi_h) = L(\phi)$ for all ϕ in the domain space and for all h in $\boldsymbol{R}^n$. It is, of course, well known what the *continuous* translation-invariant linear forms on these spaces are like; namely, they are either identically zero or a constant multiple of integration over $\boldsymbol{R}^n$.

The purpose of this paper is to announce that there exists no discontinuous translation-invariant linear form on any of the six spaces $\mathfrak{D}$, $\mathcal{E}$, $\mathcal{S}$, $\mathfrak{D}'$, $\mathcal{E}'$ or $\mathcal{S}'$. That is, integration over $\boldsymbol{R}^n$ in the spaces $\mathfrak{D}$, $\mathcal{S}$ and $\mathcal{E}'$ can be characterized (up to a multiplicative constant) simply as a translation-invariant linear form. Furthermore, we obtain this result as a simple consequence of a resolution of the first derivative of the Dirac measure δ (on the real line $\boldsymbol{R}$) into a sum of two finite differences of distributions of compact support. We state this as our main result.

THEOREM 1. *If α and β are nonzero real numbers such that α/β is irrational and not a Liouville transcendental, then there exist two (necessarily distinct) distributions S and T on $\boldsymbol{R}$, both with compact*

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supports, such that

(1) $$\delta' = S - S_\alpha + T - T_\beta,$$

and conversely.

Here S_α denotes the translate of the distribution S by the real number α, and is defined by the equation $\langle S_\alpha, \phi\rangle = \langle S, \phi_{-\alpha}\rangle$ for all test functions ϕ. Furthermore, S and T can be chosen to have order 2 (at least when α/β is a quadratic irrational) but can not have any lower order.

Note that if ϕ belongs to any of the spaces $\mathfrak{D}$, $\mathcal{E}$, $\mathcal{S}$, or their duals, then the convolution products $u \equiv \phi * S$ and $v \equiv \phi * T$ exist and belong to the same space as ϕ. Thus by convolution with ϕ formula (1) yields

(2) $$\phi' = u - u_\alpha + v - v_\beta,$$

with ϕ, u and v all in the same space. Equations (1) and (2) can be generalized to R^n (for $n \geqq 2$) by means of the tensor product of distributions. Equation (2), or its generalization to R^n, implies that the null space $\mathfrak{N}$ of a translation-invariant linear form L (on $\mathfrak{D}(R^n)$, for example) must contain the null space of integration. Consequently, there must exist a complex constant c such that $L(\phi) = c \cdot \int_{R^n} \phi(t)dt$ for all ϕ in $\mathfrak{D}(R^n)$.

The details of the proofs of Theorem 1 and related results, and the proofs of the other statements made above concerning S and T in formula (1), are to appear in J. Functional Analysis. We only indicate here the main steps in the proof of Theorem 1. According to Liouville (see [1, Theorem 191, p. 161]), if α/β is an algebraic real number of degree $\ell \geqq 2$, there exists a positive constant K such that, for all nonzero integers k,

(3) $$|1 - \exp[-2\pi i\alpha k/\beta]|^{-1} \leqq K|k|^{\ell-1}.$$

We shall consider here only the case that $\ell = 2$ (α/β is a quadratic irrational). We define an entire analytic function $\hat{S}(z)$ by the expression

$$(-z/4\pi^2)(1 - \exp[-2\pi i\beta z])^3 \sum_{-\infty}^{+\infty}{}_k' (1 - \exp[-2\pi i\alpha k/\beta])^{-1}(\beta z - k)^{-3}.$$

Then $\hat{S}$ can be shown to satisfy

$$\hat{S}(k/\beta) = (2\pi ik/\beta)(1 - \exp[-2\pi i\alpha k/\beta])^{-1},$$

for all nonzero integers k. Also the inequality (3) allows us to establish the following estimate for $\hat{S}(z)$.

(4) $$|\hat{S}(z)| \leqq C|z|(1 + |z|)e^{b|y|}, \quad \text{for all } z = x + iy,$$

for some positive constants C and b. It follows that

$$\hat{T}(z) \equiv [2\pi iz - \hat{S}(z)(1 - \exp[-2\pi i\alpha z])](1 - \exp[-2\pi i\beta z])^{-1} \tag{5}$$

is also entire and can be shown to satisfy the estimate

$$|\hat{T}(z)| \leqq B|z|(1 + |z|)e^{c|y|}, \quad \text{for all } z = x + iy, \tag{6}$$

for some constants B and c. Now the inequalities (4) and (6) imply according to the Paley-Wiener-Schwartz Theorem (see [2, Théorème XVI, p. 272]) that $\hat{S}$ and $\hat{T}$ are the Fourier transforms of two distributions S and T of compact support on the real line $\boldsymbol{R}$. But then taking inverse Fourier transforms of both sides of equation (5), after first multiplying through by the factor $(1-e^{-2\pi i\beta z})$, we obtain formula (1) of Theorem 1.

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