RECENT RESULTS IN THE FIXED POINT THEORY OF CONTINUOUS MAPS

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1. **Introduction.** Given a function $f\colon X \to X$, any question which inquires into the existence, nature and number of points $x \in X$ such that $f(x) = x$ is called fixed point theory. The assumptions on f and X range from practically none (e.g., X is a set, f a function) to quite stringent assumptions on f and X (e.g., X is a Riemannian manifold and f is an isometry). Our attention will be focused on results which require X to be a fairly reasonable space (e.g., a finite polyhedron) and f a map (=continuous function). Furthermore, we will limit our discussion to results which are not included in the expository tract [**49**] by Van der Walt (1967), which adequately covers the history of the subject from its beginning around 1910 to the early sixties.

2. **The Lefschetz theorem and local index theory.** One of the most useful tools in fixed point theory is the Lefschetz Fixed Point Theorem [**34**], [**35**], [**25**]. In its most elementary form it is simply this. Let X denote a finite polyhedron and $f\colon X \to X$ a map. Then, using the field of rationals Q as coefficients, f induces homomorphisms.

$$f_{*k}\colon H_k(X; Q) \to H_k(X; Q). \tag{1}$$

The number (it turns out to be an integer)

$$L(f) = \sum_k (-1)^k \operatorname{Trace} f_{*k} \tag{2}$$

is called the Lefschetz number of f. Then a sufficient condition for f to have at least one fixed point is that $L(f) \neq 0$. In short,

$$L(f) \neq 0 \Rightarrow f(x) = x \qquad \text{for some } x \in X. \tag{3}$$

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The converse statement for (3) is manifestly false. All one needs to look at is a polyhedron X of Euler characteristic 0 and let f denote the identity map. However, if we alter (3) slightly, making use of the fact that $L(f)$ depends only upon the homotopy class of f, the converse statement becomes more interesting (as we shall see). For this reason we formulate the Lefschetz Fixed Point Theorem as follows:

2.1. THEOREM. *Let $f:X\to X$ denote a self-map of a finite polyhedron X. If $L(f)\neq 0$, then every map homotopic to f has a fixed point.*

Another useful tool is the local form of 2.1, commonly referred to as local index theory ($L(f)$ may be thought of as a "global index"). It has a long history dating back to Hopf [**26**], and developed further by Leray [**38**], Felix Browder [**6**], [**7**], O'Neill [**44**], Bourgin [**3**] and Deleanu [**13**]. Axiomatically, the theory (following [**6**]) goes as follows.

2.2. DEFINITION. Let $\mathcal{C}$ denote a category of spaces and maps and let $A(\mathcal{C})$ denote the set of pairs (f, U), where $f:X\to X$ is a map in $\mathcal{C}$ and U is an open subset of X such that f has no fixed points on the boundary of U. Then a local index theory on $\mathcal{C}$ is a function $i:A(\mathcal{C})\to Q$ subject to the following conditions:

A1 (LOCALIZATION). If (f, U) and (g, U) belong to $A(\mathcal{C})$ and $f=g$ on $\overline{U}$, then $i(f, U)=i(g, U)$.

A2 (HOMOTOPY). If f_t is a homotopy such that $(f_t, U)\in A(\mathcal{C})$ for each t, $0\leq t\leq 1$, then $i(f_0, U)=i(f_1, U)$.

A3 (ADDITIVE). If $(f, U)\in A(\mathcal{C})$ and U contains mutually disjoint open subsets V_j, $j=1, \cdots, k$, such that f has no fixed points in $U-\bigcup_{j=1}^{k} V_j$, then

$$i(f, U) = \sum_{j=1}^{k} i(f, V_j).$$

In particular, if f has no fixed points in U, $i(f, U)=0$.

A4 (NORMALIZATION). If $f:X\to X$ belongs to $\mathcal{C}$, then

$$i(f, X) = L(f).$$

A5 (COMMUTATIVE). If the maps $f:X\to Y$, $g: Y\to X$ belong to $\mathcal{C}$ and $(gf, U)\in A(\mathcal{C})$, then

$$i(gf, U) = i(fg, g^{-1}(U)).$$

Note that A4 requires that $L(f)$ be defined for maps $f:X\to X$ in the category $\mathcal{C}$. Thus, underlying the axiomatic approach is a homology theory (with rational coefficients) H such that $H(X)$ is finitely generated for each $X\in\mathcal{C}$. In practice, Čech homology or singular homology theory are employed. Also, the existence of a local index theory on $\mathcal{C}$ implies the validity of the Lefschetz Theorem 2.1 for a map $f:X\to X$ in $(\mathcal{C})$.

The existence and uniqueness of a local index in the categories of finite polyhedra and ANR's (compact metric) is accomplished in various ways in [3], [6], [13], [38], [44]. Actually, once the existence and uniqueness of a local index for the category of finite polyhedra and maps is established it is a simple matter to extend the result to the category of ANR's using the fact that for any $\epsilon>0$, an ANR is ϵ-dominated by a finite polyhedron. A more recent development of index theory for finite dimensional ANR's is given in [14]. An excellent general reference is a forthcoming book by R. F. Brown [9]. In the case of ANR's, i takes on only integer values.

The question as to the range of validity of Theorem 2.1 is a natural one and, as already indicated, Theorem 2.1 remains valid for self-maps of ANR's (Lefschetz [36]). Lefschetz also proved 2.1 for a class of spaces called *quasi-complexes* [32]. While there are certainly quasi-complexes which are not ANR's [15], there remains the question whether every ANR is a quasi-complex. Of course, this question could be bypassed by showing that Theorem 2.1 is indeed valid for a class of spaces which contains *both* the class of compact ANR's and the Lefschetz quasi-complexes. More about this in a moment.

Simultaneously, there is also the question of finding a category $\mathcal{C}$ which properly contains the ANR's and also admits a local index theory. By altering Lefschetz's definition of a quasi-complex in significant ways, F. Browder [6] arrived at the concept of a *semicomplex*, or, more precisely, the concept of a *semicomplex structure* on a compact space X. Associated with this concept is the notion of when a map $f:X\to Y$ is a semicomplex map, where X and Y have specified semicomplex structures. If we call a category $\mathcal{C}$ of compact spaces and mappings *admissible* if each $X\in\mathcal{C}$ has a specified semicomplex structure such that all the maps in $\mathcal{C}$ become semicomplex maps, Browder showed in [6] that every admissible category $\mathcal{C}$ admits a local index theory. In addition, the Lefschetz Theorem 2.1 is valid for any space admitting a semicomplex structure. Since the category of compact metric HLC* spaces [32] and all maps is an admissible category and also contains the category of ANR's, this provided a satisfactory solution to the question which opened this paragraph [32]. However, there were still some loose ends. For example, the existence of a semicomplex structure on X implies that X is locally connected. This allows the existence of quasi-complexes which are not semicomplexes. On the other hand, the problem of whether a semicomplex is a quasi-complex brings us back to the question of whether an ANR is a quasi-complex. This left the relationship between quasi-complexes and semicomplexes rather cloudy. This problem, and others, was taken up by R. B. Thompson [46], and he succeeded in clarifying the situation. After modifying slightly Browder's definition of semicomplex (without jeopardizing Browder's results) he discovered the notion of what he called a *weak semicomplex structure* on a compact space (briefly, a *weak semicomplex*). As the terminology suggests, the concept drops some of the assumptions in the definition of a semicomplex. He then verified that the concept was strong enough to admit the validity of the Lefschetz Theorem (2.1). In addition, he proved that every quasi-complex was, in fact, a weak semicomplex and he gave necessary and sufficient conditions for a weak semicomplex to be a quasi-complex. Thus, he obtained a class of spaces wherein the Lefschetz Fixed Point Theorem was valid and at the same time contained both the quasi-complexes of Lefschetz and the semicomplexes of Browder. In summary, the category of weak semicomplexes of Thompson is appropriate for the global index, while Browder's semicomplexes appear to be the right setting for the local theory.

We might also mention that Thompson [46], [48] has shown that the category of weak semicomplexes is closed under products, suspensions, and retractions. Corresponding results also hold for semicomplexes.

3. **Nielsen classes and results of the Wecken type.** When X is a reasonable space, say an ANR (compact metric), the set $\Phi(f)$ of fixed points of $f: X \to X$ admits an equivalence relation which partitions $\Phi(f)$ into a *finite* number of subsets called the *Nielsen classes* of f [43], [50]. The relation is as follows. If x_0 and x_1 belong to $\Phi(f)$, set $x_0 \equiv x_1$ if there is a path γ from x_0 to x_1 such that γ is homotopic (with ends fixed) to $f(\gamma)$. Notice that if f is the identity map or if X is simply connected there is only one Nielsen class (assuming $\Phi(f) \neq 0$).

If F is a Nielsen class of f, we use local index theory to define the *index* of F as follows. Let U denote an open subset of f such that $F \subset U$ and $\bar{U}$ is disjoint from all the remaining Nielsen classes. Then the index $i(F)$ of F is defined by $i(F) = i(f, U)$. If $i(F) \neq 0$, F is called an *essential* Nielsen class; otherwise it is *inessential*. The number $N(f)$ of *essential* Nielsen classes is called the *Nielsen number* of f. It turns out that $N(f)$ depends only on the homotopy class of f. More precisely, a homotopy connecting f and $g: X \to X$ generates a bijection from the essential Nielsen classes of f to the essential Nielsen classes of f with corresponding classes having the *same* index. Notice that

$N(f)$ is a *lower bound* for the number of fixed points of any map homotopic to f.

It is worthwhile noting here that $N(f)=0$ always implies $L(f)=0$ because of the properties of the local index theory. In particular, if $F_1, \cdots, F_k$ are the Nielsen classes of f, then

$$L(f) = \sum_{i=1}^{k} i(F_i) \tag{1}$$

so that if all the F_i are inessential, $L(f)=0$. On the other hand, there are examples of *manifolds* (in all dimensions), due to D. McCord [**41**], which admit homeomorphisms f such that $L(f)=0$ and yet $N(f)\geqq 2$. One obvious sufficient condition for $L(f)=0$ to imply $N(f)=0$ is to require X to be simply connected. A more interesting condition is due to Jiang [**28**] who associates with each $f: X\to X$ a subgroup $J(f)$ of the fundamental group $\pi_1(X)$ as follows. Choose a base point $x_0\in X$ and let e: $\mathrm{Map}(X, X)\to X$ denote the evaluation map at x_0. Using f as the base point in $\mathrm{Map}(X, X)$, e induces

$$e_\#: \pi_1(\mathrm{Map}(X, X), f) \to \pi_1(X, f(x_0)) \tag{2}$$

the image of $e_\#$ is called the *Jiang subgroup* $J(f)$ of f. It is independent of the base point (assuming X is 0-connected) and depends only on the homotopy class of f. Alternatively, $J(f)$ is the subgroup of $\pi_1(X)$ generated by *cyclic homotopies* based at f.

3.1. THEOREM (JIANG [**28**]). *If $J(f)=\pi_1(X)$, then $L(f)=0$ implies $N(f)=0$.*

Actually Theorem 3.1 is a corollary of a more general result that states that if $J(f)=\pi_1(X)$, then all the Nielsen classes of f have the *same* index. Since it is a tedious procedure to verify the Jiang condition for a given map f it should be remarked that if we let $J(X)$ denote J (identity) that $J(X)\subseteq J(f)$. Thus, the condition $J(X)=\pi_1(X)$ is sufficient for the validity of Theorem 3.1 for all maps f. In addition to the trivial observation that $J(X)=\pi_1(X)$ when X is simply connected, it should be noted that $J(X)=\pi_1(X)$ wherever X is an H-space [**28**].

In the light of Theorem 3.1, it is natural to study $N(f)$ under the assumption that $L(f)\neq 0$. Results on estimating $N(f)$ under the hypothesis, as well as related results have been obtained by Jiang [**28**], Barnier [**1**], Gottlieb [**22**], and Brooks and Brown [**4**].

We now turn to the problem of determining when the Nielsen number $N(f)$ is the best possible lower bound on the number of fixed points that a map homotopic to f can have. Since the fundamental work on this problem is due to Wecken [**50**], we adopt the following definition.

3.2. DEFINITION. An ANR X (compact metric) is called a *Wecken space* if for any map $f: X \to X$, there is a map g homotopic to f such that g has precisely $N(f)$ fixed points.

Wecken [**50**] proved

3.3. THEOREM. *Every (connected) finite polyhedron K with the property that* $\mathrm{St}\sigma - \sigma$ *is connected for every* 0 *or* 1*-simplex of K is a Wecken space.*

In the above $\mathrm{St}\sigma$ denotes the open star in K of the open simplex σ. Recently, Shi-Gen Hua [**45**] improved Wecken's result as follows.

3.4. THEOREM. *Every (connected) finite polyhedron which contains a 3-simplex and has the property that $\partial \mathrm{St} v$ (∂ = boundary) is connected for every vertex v, is a Wecken space.*

Thus, every triangulable manifold of dimension $\geqq 3$ is a Wecken space. In this regard, R. F. Brown [**10**], using some earlier results of Wier [**51**], has shown that every topological manifold (possibly with boundary) is a Wecken space.

The next theorem is just an observation, but it is the key to the converse of the Lefschetz Fixed Point Theorem 2.1.

3.5. THEOREM. *Let X be a Wecken space satisfying the Jiang condition $J(X) = \pi_1(X)$. If $f: X \to X$ has $L(f) = 0$, f is homotopic to a map g which is fixed point free.*

We may now state the Lefschetz Theorem 2.1 along with its converse.

3.6. THEOREM. *Let $f: X \to X$ denote a self-map of a Wecken space satisfying the Jiang condition $J(X) = \pi_1(X)$. Then $L(f) \neq 0$ if, and only if, every map homotopic to f has a fixed point.*

Notice that the previously mentioned examples of D. McCord [**41**] show that the Jiang condition cannot be dropped in 3.6.

A representative corollary of Theorem 3.5 (using R. F. Brown's result [**10**]) is the following

3.7. COROLLARY. *Let $f: M \to M$ denote a self-map of a compact, simply connected, topological manifold of dimension $\geqq 3$. If $L(f) = 0$ then f is homotopic to a map g which is fixed point free.*

This corollary also follows from a result of F. B. Fuller [**20**] proved for triangulated manifolds and extended to topological manifolds in [**17**].

3.8. THEOREM. *Let X denote a space dominated by a finite n-polyhedron such that $H^n(X; \mathbf{Z})$ is torsion free. Let M be a compact topological n-manifold which is simply connected and let $f, f': X \to M$ be two given maps. Then, f is homotopic to $g: X \to M$ such that g and f' are coincidence free if, and only if, the (rational) Lefschetz coincidence class $L(f, f') = 0$.*

The proof of Theorem 3.8 is based on an obstruction theory argument. Corollary 3.7 follows by letting $X = M$, $f' =$ identity, and observing that $L(f, \mathrm{id}) = L(f)\mu$, where μ is a generator of $H^n(M, \mathbf{Z})$.

As a special case of deforming a map $f: X \to X$ to obtain a new map g which is fixed point free we have the problem of deforming the identity map to a fixed point free map. Clearly, a necessary condition is that $\chi(X) = L(\mathrm{id}) = 0$, where $\chi =$ Euler characteristic. For compact differentiable manifolds, this is just the classical problem of finding nonzero vector fields, and $\chi = 0$ is the classical necessary and sufficient condition given by Hopf [27].

The concept of nonzero vector field has its analogue in a topological manifold M using Nash's tangent space of paths [42]. Briefly, a nonzero *path* field is a map $\sigma: M \to M^I$ such that $\sigma(x)$ is a path which starts at x and never returns to x. Obviously, if such a map σ exists, then $\sigma(1)$ is a fixed point free map homotopic to identity. R. F. Brown [8] proved that $\chi(M) = 0$ is necessary and sufficient for the existence of a nonzero path field on M and the result was extended to topological manifolds with boundary in [11]. It is interesting to note that in the case of differentiable manifold a nonzero vector field implies the existence of a nonzero path field σ such that $\sigma(x)$ is a simple path (no self-intersections) for each x. It is not known whether there is a corresponding result for topological manifolds.

Recall that in the case of the identity map $\mathrm{id}: X \to X$ there is only one Nielson class. Thus if $\chi(X) = L(\mathrm{id}) = 0$ and X is Wecken space, e.g., a polyhedron of dimension $\geqq 3$ with $\partial \mathrm{St} v$ connected for any vertex v, then, since $N(\mathrm{id}) = 0$, there is always a map $f \sim \mathrm{id}$, with f fixed point free.

3.9. THEOREM. *If X is a Wecken space, then X admits a fixed point free map homotopic to the identity if, and only if, $\chi(X) = 0$.*

Curiously, the identity map is easier to deform than an arbitrary map as the following result of Wecken [50], [45] shows.

3.10. THEOREM. *Let X denote a finite polyhedron with the property that each maximal simplex has dimension $\geqq 2$ and X is strongly connected in the sense that given any two 2-simplexes σ and τ there is a chain of 2-simplexes $\sigma_1, \cdots, \sigma_k$ such that $\sigma = \sigma_1$, $\tau = \sigma_k$ and σ_i and σ_{i+1} share* *a common face of dimension* 1, $i=1, \cdots, k-1$. *Then, the identity map may be deformed into a map* $f: X \to X$ *which has no fixed points if* $\chi(X)=0$ *or one fixed point if* $\chi(X) \neq 0$.

It is easy to show that a finite polyhedron X satisfies the hypotheses in Theorem 3.10 if, and only if, no finite subset of X separates X.

3.11. COROLLARY. *If X is a finite polyhedron such that no finite subset of X separates X, then X admits a fixed point free map homotopic to the identity if, and only if,* $\chi(X)=0$.

A polyhedron X satisfying the hypotheses of Theorem 3.10 need not be a Wecken space. On the other hand it is a simple matter to show a polyhedron satisfying the hypotheses of Shi's Theorem 3.4 cannot be separated by a finite subset. In this sense, Corollary 3.11 is stronger than Theorem 3.9.

4. The fixed point property. A space X has the fixed point property (f.p.p.) if *every* map $f: X \to X$ has a fixed point. We will assume throughout this section that X is no more general than a connected compact metric ANR. The results of the previous section immediately imply the following theorems:

4.1. THEOREM. *If X is a Wecken space, X has f.p.p. if, and only if, $N(f) \neq 0$ for every map $f: X \to X$.*

4.2. THEOREM. *If X is a Wecken space satisfying the Jiang condition $J(X)=\pi_1(X)$, then X has f.p.p. if, and only if, $L(f) \neq 0$ for every map $f: X \to X$.*

The usual method for showing that a space X has f.p.p. is to show that $L(f) \neq 0$ for every map f. If X satisfies the Shi condition (namely, X is a polyhedron of dimension $\geqq 3$ and $\partial \mathrm{St} v$ is connected for every vertex v) and is also simply connected, then Theorem 4.2 says that any other method is equivalent to this method.

There are times when it is convenient to use fields as coefficients other than the rationals Q. If Λ is any field and $f: X \to X$, we have induced homomorphisms $f_{*k}: H_k(X; \Lambda) \to H_k(X; \Lambda)$ and the Lefschetz number over Λ is defined, just as in the rational case, by

$$L(f; \Lambda) = \sum_k (-1)^k \operatorname{Trace} f_{*k}. \tag{1}$$

We also have the Lefschetz Theorem:

$$f \text{ fixed point free} \Rightarrow L(f; \Lambda) = 0. \tag{2}$$

Thus, to show that X has f.p.p., it suffices to show that $L(f, \Lambda) \neq 0$

for every map $f: X \to X$, where Λ is any field (which might vary with f). We might note too that we may use cohomology with coefficients in Λ to compute $L(f; \Lambda)$.

We can illustrate with simple examples. First consider complex projective space CP^n, n even. If $\alpha \in H^2(CP^n; Q)$ is a generator and $f: CP^n \to CP^n$ is a map, $f^*(\alpha) = a\alpha$. Therefore, using the ring structure on $H^*(CP^n; Q)$ we obtain

$$L(f) = 1 + a + a^2 + \cdots + a^n. \tag{3}$$

Now $L(f) \neq 0$ since the cyclotomic polynomial $1 + x + x^2 + \cdots + x^n$ has no real roots when n is even. We conclude that CP^n has f.p.p. for n even. The argument using Z_2 is simpler. If β generates $H^2(CP^n, Z_2)$ and $f^*(\beta) = b\beta$, then

$$L(f, Z_2) = 1 + b + b^2 + \cdots + b^n. \tag{4}$$

Since n is even, $L(f, Z_2) = 1$ for $b = 0, 1$. Incidentally, this argument works just as well for real projective space RP^n and quaternionic projective space HP^n, n even.

Now consider the suspension SCP^n for n even. If $1, \beta, \beta^2, \cdots, \beta^n$ are the nontrivial homogeneous elements of $H^*(CP^n; Z_2)$, let $1, S\beta, \cdots, S\beta^n$ denote the corresponding elements of $H^*(SCP^n; Z_2)$. It is a simple matter to check that $Sq^2\beta = \beta^2$, $Sq^2\beta^3 = \beta^4, \cdots$, and hence $Sq^2S\beta = S\beta^2$, $Sq^2S\beta^3 = S\beta^4, \cdots$. Hence, if $f: SCP^n \to SCP^n$ is a map and $f^*(S\beta^i) = b_iS\beta^i$, we have $b_1 = b_2$, $b_3 = b_4, \cdots, b_{n-1} = b_n$. Thus $L(f, Z_2) = 1$ and SCP^n has f.p.p. for n even.

A few years ago I was struck by the fact that for all the examples of polyhedra that I knew had f.p.p., it sufficed to argue using Z_2 coefficients, i.e. one showed that $L(f, Z_2) = 1$ for every self-map f. This observation led to the following question.

QUESTION A. Does there exist a polyhedron X with f.p.p. which admits a self-map f such that $L(f)$ is an even integer?

We propose now to investigate what the implications of an affirmative answer to Question A are, adding the assumption that X is also simply connected.

We will consider the following category $\mathfrak{F}$. The *objects* of $\mathfrak{F}$ are based maps $f: (X, x_0) \to (X, x_0)$ where X is a compact, simply connected, triangulable space with f.p.p. A *morphism* in $\mathfrak{F}$, say $\phi: f \to f'$, is a map where

$$\begin{array}{ccc} (X, x_0) & \xrightarrow{f} & (X, x_0) \\ \phi \downarrow & & \downarrow \phi \\ (Y, y_0) & \xrightarrow{f'} & (Y, y_0) \end{array} \tag{5}$$

is a commutative diagram. Notice that if ϕ is an equivalence in $\mathfrak{F}$, ϕ is a homeomorphism such that $\phi f = f'\phi$. Using the wedge operation

$$f \vee g\colon (X \vee Y, (x_0, y_0)) \to (X \vee Y, (x_0, y_0)) \tag{6}$$

where $f\colon (X, x_0) \to (X, x_0)$, $g\colon (Y, y_0) \to (Y, y_0)$, the category $\mathfrak{F}$ admits a "sum" operation. Here we make use of the simple fact that the wedge of two spaces with f.p.p. also has f.p.p. If we let $[\mathfrak{F}]$ denote the isomorphy classes of $\mathfrak{F}$ under equivalence, this wedge operation makes $[\mathfrak{F}]$ into an abelian semigroup with zero. The zero element corresponds to a point-map $x_0 \to x_0$.

If $f \in \mathfrak{F}$, we let $\bar{L}(f)$ denote the *reduced* Lefschetz number of f, i.e.,

$$\bar{L}(f) = \sum_{k \geqq 1} (-1)^k \operatorname{Trace} f_{k*}. \tag{7}$$

Of course, $\bar{L}(f) = L(f) - 1$ since we are dealing only with connected spaces. It is immediate that

$$\bar{L}(f \vee g) = \bar{L}(f) + \bar{L}(g). \tag{8}$$

Furthermore, if f and g are equivalent in $\mathfrak{F}$, $\bar{L}(f) = \bar{L}(g)$. This means that $\bar{L}$ induces a homomorphism

$$\bar{L}\colon [\mathfrak{F}] \to \boldsymbol{Z}. \tag{9}$$

We wish to investigate what the consequences of the following hypothesis are.

HYPOTHESIS B. There is a simply connected polyhedron X which admits a map $f\colon X \to X$ such that $L(f)$ is even.

4.3. THEOREM. *Hypothesis* B *implies that* $\bar{L}$ *in* (9) *is surjective.*

PROOF. If we let $\tilde{\chi}$ denote the reduced Euler characteristic, then

$$\tilde{\chi}(CP^n) = n, \qquad \tilde{\chi}(SCP^n) = -n. \tag{10}$$

Since CP^n and SCP^n have f.p.p. for n even, it follows that the image of $\bar{L}$ contains the even integers. Hypothesis B asserts that the image of $\bar{L}$ contains some odd integer. This implies that $\bar{L}$ is surjective.

If we let $[\mathfrak{F}] \xrightarrow{\eta} G[\mathfrak{F}]$ denote the corresponding Grothendieck group of $[\mathfrak{F}]$, then Theorem 4.3 implies that $G[\mathfrak{F}]$ contains $\boldsymbol{Z}$ as a direct factor. A more interesting corollary is the following.

4.4. COROLLARY. *Hypothesis* B *implies there exists a simply connected polyhedron* X *with f.p.p. which admits a self-map* f *such that* $L(f) = 0$.

Thus Hypothesis B implies the existence of a simply connected counterexample X to Theorem 4.2. It is easy to see that X (assuming $\dim X \geqq 3$) must be a nontrivial wedge of two subpolyhedra.

The following simple lemma implies additional interesting consequences of Hypothesis B.

4.5. LEMMA. *Given maps* $f\colon X \to X$, $g\colon Y \to Y$ *we have* $L(f\times g) = L(f)L(g)$, $\bar{L}(Sf) = -\bar{L}(f)$, $\bar{L}(f\wedge g) = \bar{L}(f)\bar{L}(g)$ *and* $\bar{L}(f \circ g) = -\bar{L}(f)\bar{L}(g)$, *where* $S =$ *suspension*, $\wedge =$ *smash product, and* $\circ =$ *join.*

4.6. COROLLARY. *Hypothesis* B *implies that for each of the following constructions C, there exists a simply connected polyhedron X with f.p.p. such that* $C(X)$ *admits a self-map f with* $L(f) = 0$:

(a) $C(X) = X \times I$,
(b) $C(X) = X \times X$,
(c) $C(X) = SX$,
(d) $C(X) = X \wedge X$,
(e) $C(X) = X \circ X$.

PROOF. For (a) and (b), choose $g \in \mathfrak{F}$ such that $L(g) = 0$. Then $f = g \times \mathrm{id}$ works in both cases. With this same g, $L(g \circ g) = 0$ so that, $f = g \circ g$ works for (e). For (c), choose $g \in \mathfrak{F}$ such that $L(g) = 2$. Then $f = Sg$ has $L(f) = 2 - L(g) = 0$ and this f works for (c). Finally, for (d) we will need a space X which admits *two* self-maps g_1 and g_2 such that $L(g_1) = 0$ and $L(g_2) = 2$. Then $\bar{L}(g_1 \wedge g_2) = \bar{L}(g_1)\bar{L}(g_2) = -1$ and hence $f = g_1 \wedge g_2$ will do the job. This is easily accomplished by choosing $X = A \vee B$, where $g_1'\colon A \to A$, $g_2'\colon B \to B$ are chosen in so that $L(g_1') = 0$, $L(g_2') = 2$.

The next lemma makes things even more interesting.

4.7. LEMMA. *If X is a simply connected polyhedron* ($\dim X \geqq 2$), *then for each of the constructions C in Corollary* 4.6, $C(X)$ *is simply connected and satisfies the Shi condition* ($\dim C(X) \geqq 3$ *and in some triangulation of* $C(X)$, $\partial \mathrm{St}v$ *is connected for every vertex v*). *The smash product requires taking as base point* (x_0, y_0) *where* x_0 *and* y_0 *are not separating points.*

4.8. COROLLARY. *Hypothesis* B *implies that for each of the constructions C in Corollary* 4.6, *there exists a simply connected polyhedron X with f.p.p. such that* $C(X)$ *fails to have f.p.p.*

Summarizing, Hypothesis B implies

4.9. THEOREM. *In the category of simply connected polyhedra, the fixed point property is not invariant under cartesian products, cartesian product with I, suspension, smash product, join or homotopy type.*

Results corresponding to those in Theorem 4.9 in more general categories (e.g. compact connected metric spaces) are known (Kinoshita [30], Connell [12], Knill [31]). The examples involved are not locally contractible.

W. Lopez [38] was the first to verify Hypothesis B. He considered the space

$$X = CP^2 \cup S_1 \times S_2 \cup CP^4 \tag{11}$$

where S_1 and S_2 are 2-spheres with S_1 identified with $CP^1 \subset CP^2$ and S_2 identified with $CP^1 \subset CP^4$. A simple computation shows that $\chi(X)=8$. Using the cohomology ring of X, he was able to show that $L(f)\neq 0$ for every self-map f of X so that X has f.p.p. This example gives more than required in Hypothesis B since the identity map id: $X\to X$ does the job. Notice the $X \vee SCP^8$ has Euler characteristic 0 so that this wedge is a specific example verifying Corollary 4.4. In addition to Theorem 4.9, we mention two other anomalies. First of all, $X \cup_I SCP^3 = A$ does not have f.p.p., where $\cup_I$ denotes union along an edge. This is because A is a simply connected Wecken space. On the other hand, we can attach a two cell D^2 to $X \vee SCP^8$ along an edge in two different ways, preserving f.p.p. one way (Figure 1a) and destroying f.p.p. it with the other (Figure 1b).

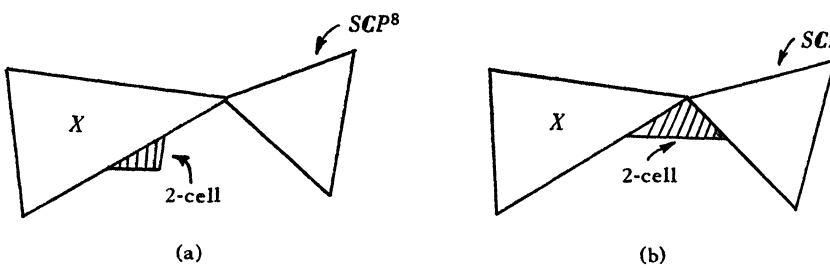

FIGURE 1

REMARK. Outside of the ANR category, R. H. Bing [2] has constructed an example of a one dimensional continuum C with f.p.p. such that adding a 2-cell along a 1-cell destroys f.p.p.

After the publication of Lopez's example, G. Bredon supplied another, namely quaternionic projective 3-space, HP^3 which has Euler characteristic 4. This was surprising since the corresponding real and complex projective spaces do not have f.p.p. Making simple use of the Steenrod algebra $\mathcal{A}(3)$, Bredon showed that HP^3 does not admit a self-map f such that $f^*(\alpha)=-\alpha$ where $\alpha\in H^4(HP^3)$ is a generator. Since

$$(12) \qquad L(f) = 1 + a + a^2 + a^3$$

if $f^*(\alpha) = a\alpha$, $L(f) \neq 0$ for any map f and HP^3 has f.p.p. A similar argument shows that all the quaternionic projective spaces HP^k have f.p.p. for $k \geqq 2$.

We may summarize by stating that in the category of simply connected polyhedra, the f.p.p. behaves badly with regard to geometric constructions except for the wedge operation. This naturally leads us to study f.p.p. in more restrictive categories. We have in mind four possibilities;

(a) Category $\mathcal{S}$: Polyhedra satisfying the Shi condition.

(b) Category $\mathcal{S}_0$: Spaces in Category $\mathcal{S}$ Satisfying the Jiang condition.

(c) Category $\mathfrak{M}$: Compact topological manifolds, $\dim \geqq 3$.

(d) Category $\mathfrak{M}_0$: Spaces in Category $\mathfrak{M}$ satisfying the Jiang condition.

Fortunately, there are a few positive results. The following simple theorem has interesting consequences.

4.10. THEOREM. *If $f: X \to Y$ is a map where Y belongs to $\mathcal{S}$ or $\mathfrak{M}$ and has f.p.p., then the mapping cylinder $M(f)$ has f.p.p. (recall X is a compact metric ANR).*

REMARK. If X belongs $\mathcal{S}_0$ (or $\mathfrak{M}_0$) and has f.p.p., then using results of Jiang [26], it follows that X is simply connected.

4.11. COROLLARY. *If X has the same homotopy type as a space Y belonging to $\mathcal{S}$ or $\mathfrak{M}$ which has f.p.p., then X has f.p.p.*

4.12. COROLLARY. *If X belongs to $\mathcal{S}$ or $\mathfrak{M}$ and has f.p.p., then $X \times I$ has f.p.p.*

4.13. COROLLARY. *If X belongs to $\mathcal{S}$ or $\mathfrak{M}$ and has f.p.p., then $Z = X \cup Y$ has f.p.p. when Y and $X \cap Y$ are AR's.*

REMARK. In Corollary 4.11 it is only necessary that X is *dominated by* Y.

The most interesting positive result would be the product theorem in the categories $\mathcal{S}_0$, $\mathcal{S}$, $\mathfrak{M}_0$, $\mathfrak{M}$.

4.14. QUESTION. If X and Y belong to $\mathcal{S}_0$ (or $\mathcal{S}$, $\mathfrak{M}_0$, $\mathfrak{M}$) and both have f.p.p., does $X \times Y$ have f.p.p.?

It is difficult to conjecture whether the answer to Question 4.14 is affirmative or negative. However, the following examples illustrate that the f.p.p. behaves badly even in the more restrictive categories and this makes one suspicious of an affirmative answer.

Let

$$K = HP^4 \cup_I SHP^3 \tag{13}$$

where $\cup_I$ = union along an edge, and

$$M = RP^8 \# HP^2 \tag{14}$$

where # denotes connected sum. Both K and M can be shown to have f.p.p. [**18**].

The polyhedron K is simply connected and satisfies the Shi condition so that $K \in \mathcal{S}_0$. However $\chi(K) = 2$ and hence $\chi(SK) = 0$ and $\chi(K \circ K) = 0$. By Corollary 3.11 neither SK nor $K \circ K$ have f.p.p. On the other hand $K \times I$ has f.p.p.

4.16. Theorem. *The f.p.p. is not invariant under suspension and join in the category* $\mathcal{S}_0$.

Except for the fact that M is not simply connected (it does not satisfy the Jiang condition), M behaves like K. First of all, the rational cohomology of M is that of a 4-sphere so that $\chi(M) = 2$. This implies $\chi(SM) = 0 = \chi(M \circ M)$. Again, using Corollary **3.11**, we see that SM and $M \circ M$ fail to have f.p.p. On the other hand, both $M \times I$ and $M \times M$ have f.p.p. [**18**].

4.17. Theorem. *The f.p.p. is not invariant under suspension and join in the category* $\mathfrak{M}$.

We are left with some more questions.

4.18. Question. What is the behavior of f.p.p. with regard to suspension and join in the category of simply connected manifolds?

4.19. Question. What is the behavior of f.p.p. under smash product in $\mathcal{S}$, $\mathcal{S}_0$, $\mathfrak{M}$ and $\mathfrak{M}_0$.

We close this section with one more question. This pertains to the necessity of assuming the Jiang condition in Theorem 4.2.

4.20. Question. Does there exist a space X in $\mathcal{S}$ (or $\mathfrak{M}$) with f.p.p. which admits a self-map f such that $L(f) = 0$?

5. **Fixed points for iterates.** Often it is useful to know whether some iterate f^k of $f: X \to X$ has a fixed point. Representative results in this direction has been obtained by F. Browder [**5**] and F. B. Fuller [**19**]. More recently a simple method for determining whether some iterate of f has a fixed point has evolved which is worthy of mention. The tool is contained in a paper of Kelley and Spanier [**29**] and also in the work of Hajek [**23**]. Both Halpern [**24**] and Hajek [**23**] used this tool to obtain results in this direction and to provide an alterna-

tive proof for Fuller's result [19] that if $f: X \to X$ is a homeomorphism and $\chi(X) \neq 0$, then some iterate of f has a fixed point. In this section we will give the details of the method and illustrate the technique with a proof of Fuller's generalization [21] of his aforementioned result.

Let $\boldsymbol{Q}$ denote the field of rationals and $\boldsymbol{Q}(\lambda)$ the field of rational functions over $\boldsymbol{Q}$. Let $e: \boldsymbol{Q}(\lambda) \to \boldsymbol{Q}(\lambda)$ denote the multiplicative automorphism given by $e(f(\lambda)) = f(1/\lambda)$ and $q: \boldsymbol{Q}(\lambda) - 0 \to \boldsymbol{Q}(\lambda)$ the formal logarithmic derivative defined by $q(r(\lambda)) = r'(\lambda)/r(\lambda)$. We let

$$D = qe: \boldsymbol{Q}(\lambda) - 0 \to \boldsymbol{Q}(\lambda) \tag{1}$$

and note that D takes products into sums and $D(f) = 0$ if, and only if, f is constant.

Each nonzero element of $\boldsymbol{Q}(\lambda)$ has a unique formal expansion of the form

$$\lambda^m \left(\sum_{j=0}^{\infty} a_j \lambda^j \right), \qquad a_0 \neq 0, \qquad m \text{ an integer.} \tag{2}$$

The basic idea is to look at the series form (2) of the image under D of the characteristic polynomial of a linear operator.

Let $T: A \to A$ denote a linear operator where A is a vector space over $\boldsymbol{Q}$ of finite dimension m and $f(\lambda) = |\lambda I - T|$ the corresponding characteristic polynomial. Working over the complex field $\mathbf{C}$, we may write

$$f(\lambda) = (\lambda - e_1) \cdots (\lambda - e_m) \tag{3}$$

and hence

$$r(\lambda) = f(1/\lambda) = \lambda^{-m}(1 - \lambda e_1) \cdots (1 - \lambda e_m), \tag{4}$$

$$r'(\lambda) = \left(- \frac{m}{\lambda} \right) r(\lambda) - \sum_{j=1}^{m} \left(\frac{e_j}{1 - \lambda e_j} \right) r(\lambda). \tag{5}$$

Therefore

$$D(f(\lambda)) = \frac{r'(\lambda)}{r(\lambda)} = - \frac{m}{\lambda} - \sum_{j=1}^{m} \frac{e_j}{1 - \lambda e_j} \,. \tag{6}$$

If we expand (6) using

$$\frac{e_j}{1 - \lambda e_j} = e_j(1 + \lambda e_j + \lambda^2 e_j^2 + \cdots) \tag{7}$$

we obtain

$$D[f(\lambda)] = -\frac{m}{\lambda} - \sum_{k=1}^{\infty} (e_1^k + \cdots + e_m^k)\lambda^{k-1}. \tag{8}$$

Using the Jordan canonical form for T, we recall that

$$\text{Trace } (T^k) = \text{Tr}(T^k) = e_1^k + \cdots + e_m^k. \tag{9}$$

Hence we arrive at

5.1. LEMMA. *If* $f(\lambda) = |\lambda I - T|$, *where* $T: A \to A$ *is a linear operator and* dim $A = m$, *then*

$$D[F(\lambda)] = -\sum_{k=0}^{\infty} \text{Tr}(T^k)\lambda^{k-1}. \tag{10}$$

Now, if $\phi: X \to X$ is a map where X is a compact metric ANR, we call

$$f(\lambda) = \prod_q |\lambda I - \phi_{*2q}| \Big/ \prod_q |\lambda I - \phi_{*2q+1}| \tag{11}$$

the *characteristic function* for ϕ where

$$\phi_{*q}: H_q(X) \to H_q(X) \tag{12}$$

are the induced homology endomorphisms (rational coefficients). Lemma 5.1 implies

5.2. LEMMA. *If* $f(\lambda)$ *is the characteristic function for* $\phi: X \to X$, *then*

$$D[f(\lambda)] = -\sum_{k=0}^{\infty} L(\phi^k)\lambda^{k-1}. \tag{13}$$

5.3. DEFINITION. If $f(\lambda)$ is the characteristic function for $\phi: X \to X$ and

$$r(\lambda) = f\left(\frac{1}{\lambda}\right) = \lambda^{-\chi}\left(1 + \sum_{j=1}^{\infty} a_j\lambda^j\right) \tag{14}$$

and call the coefficients a_j, $j \geqq 1$, the *canonical coefficients* of ϕ. Here $\chi = \chi(X)$.

Now if one uses (14) to compute $r'(\lambda)/r(\lambda)$ and equates the result with (13) we see that

$$a_j = 0 \quad \text{for } 1 \leqq j < s \Leftrightarrow L(\phi^j) = 0 \quad \text{for } 1 \leqq j < s. \tag{15}$$

Thus we arrive at the following tool.

5.4. THEOREM. *If* ϕ *is a self-map of a compact metric* ANR X *and* *one of its canonical coefficients* $a_k \neq 0$, *then one of the iterates* $\phi, \phi^2, \cdots, \phi^k$ *has a fixed point.*

It is easy to see that $a_1 = -L(\phi)$ so that the above theorem may be regarded as a generalization of the Lefschetz Fixed Point Theorem.

Before we use Theorem 5.4 to prove Fuller's Theorem [21], we need to recall a simple fact. If $T: A \to A$ is a linear operator on a vector space A over Q of dimension m, let ρ denote the minimum rank of all the iterates of T. If we write $|\lambda I - T| = (\lambda - e_1) \cdots (\lambda - e_m)$, then ρ is just the number of nonzero e_j's (multiplicity allowed).

If $\phi: X \to X$ is a map, let ρ_q denote the minimum rank to which the iterates of ϕ_{*q} descend, and set

$$F(\phi) = \sum_{i=0}^{\infty} (-1)^q \rho_q. \tag{16}$$

5.5. THEOREM (F. B. FULLER). *If* ϕ *is a self-map of an* ANR X *and the Fuller index* $F(\phi) \neq 0$ *then one of the iterates* ϕ^k *has a fixed point where*

$$k \leqq \max\left[\sum_q \rho_{2q}, \sum_q \rho_{2q+1}\right].$$

PROOF. Let us write the characteristic function $f(\lambda)$ for ϕ in the form

$$f(\lambda) = n(\lambda)/d(\lambda) \tag{17}$$

where

$$n(\lambda) = \prod_q |\lambda I - \phi_{*2q}| = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n \tag{18}$$

and

$$d(\lambda) = \prod_q |\lambda I - \phi_{*2q+1}| = \lambda^m + d_1\lambda^{m-1} + \cdots + d_m. \tag{19}$$

If we let

$$s = \sum_q \rho_{2q}, \qquad t = \sum_q \rho_{2q+1}$$

then

$$s = \text{maximum } j \text{ such that } \quad c_j \neq 0,$$
$$t = \text{maximum } j \text{ such that } \quad d_j \neq 0.$$

Furthermore,

$$r(\lambda) = f\left(\frac{1}{\lambda}\right) = \lambda^{-x} \frac{(1 + c_1\lambda + \cdots + c_n\lambda^n)}{(1 + d_1\lambda + \cdots + d_m\lambda^m)}. \tag{20}$$

Therefore, in terms of the canonical coefficients a_j, we have

$$\begin{aligned}(21)\quad & 1 + c_1\lambda + \cdots + c_n\lambda^n \\ & = (1 + d_1\lambda + \cdots + d_m\lambda^m)(1 + a_1\lambda + a_2\lambda^2 + \cdots)\end{aligned}$$

and equating coefficients

$$c_k = a_k + a_{k-1}d_1 + \cdots + a_1d_{k-1} + d_k. \tag{22}$$

If $s>t$, set $k=s$ in (22) and we see that not all of the canonical coefficients $a_1, \cdots, a_s$ can vanish since $d_s=0$. Therefore, for some $k\leqq s$, ϕ^k has a fixed point. If $s<t$, we set $k=t$ in (22) and obtain for some $k\leqq t$, ϕ^k has a fixed point.

5.6. COROLLARY (FULLER). *If $\phi: X\to X$ is a homeomorphism and $\chi(X)\neq 0$, then some iterate of ϕ has a fixed point.*

5.7. COROLLARY (HALPERN, HAJEK). *If $\phi: X\to X$ is a map and the rational homology of X vanishes in odd dimensions, then some iterate of ϕ has a fixed point.*

5.8. REMARKS. Theorem 5.4 requires only that the Lefschetz Fixed Point Theorem be valid for the space X. It therefore applies to weak semicomplexes [**46**], for example. The proof of Theorem 5.5 is a variation of Halpern's proof of Corollary 5.6.

ADDED IN PROOF. The example $\boldsymbol{RP}^8 \# \boldsymbol{HP}^2$ in §4 which was used to prove Theorem 4.17 is incorrect. However, the manifold $\boldsymbol{HP}^2 \# \boldsymbol{HP}^2$ has f.p.p. and may be used instead [**18**]. Since $\boldsymbol{HP}^2 \# \boldsymbol{HP}^2$ is simply connected we also have an answer to Question 4.18. The argument that a self-map f of $\boldsymbol{RP}^8 \# \boldsymbol{HP}^2$ has fixed points requires that f does not kill the nontrival element of $\pi_1(\boldsymbol{RP}^8 \# \boldsymbol{HP}^2)$. Furthermore, as pointed out to me by Richard Goldstein, $\boldsymbol{RP}^8 \# \boldsymbol{HP}^2$ admits fixed point free maps.

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