NOTES TOWARDS THE CONSTRUCTION OF NONLINEAR RELATIVISTIC QUANTUM FIELDS. III: PROPERTIES OF THE C^*-DYNAMICS FOR A CERTAIN CLASS OF INTERACTIONS

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Communicated by Murray Protter, September 17, 1969

1. This note treats the C^*-dynamics of positive-energy symmetric (or 'Bose-Einstein') quantum fields in continuation of [1]. The temporal development of the systems being considered is given by a one-parameter group of automorphisms of a C^*-algebra, which in general are not unitarily implemented, but may by a process of localization be reduced to the consideration of a complex of putative one-parameter unitary groups. Each such group is to be generated by an operator H' which is formally given as $H+V$, where each of H and V may be formulated as a selfadjoint operator in Hilbert space, but whose sum is *a priori* ill-defined as such because of the singular nature of V in relation to H.

In [1, I] a theory of renormalized products of quantum fields was initiated which served as a basis for the treatment of the operators V of concrete interest. It followed that for a certain class of relativistic cases:

(a) $H+V$ is densely defined and has a selfadjoint extension H';

(b) the associated complex of one-parameter unitary groups corresponds to a C^*-automorphism group provided the Lie formula: $e^{itH'}=\lim_n(e^{itH/n}e^{itV/n})^n$ is applicable (as is the case e.g. if H' is unique, by a theorem of Trotter). In the present note, by making a natural use of mild particularities of the operators in question, a selfadjoint extension H' is constructed which has the modified property, sufficient for the construction of an appropriate C^*-automorphism group, that $e^{itH'}=\lim_m\lim_n(e^{itH/n}e^{itf_m(V)/n})^n$, if $\{f_m\}$ is any sequence of real functions of compact support on R^1 such that $f_m(\lambda)\rightarrow\lambda$ and $|f_m(\lambda)|\leqq|\lambda|$; and this operator has in addition many other relevant properties. The treatment is quite general, and apart from the finiteness of the moments of V and e^{-V}, and the nonvanishing of the 'mass,' makes no significant assumptions.

2. While the chief goal of the present notes is the construction of nonlinear relativistic quantum fields, it is of mathematical as well as potentially of physical interest to explore other possible formulations of space-time. In partially colloquial terms, the situation may

AMS Subject Classifications. Primary 4748, 8146; Secondary 2846, 8135.

Key Words and Phrases. Renormalized power, generalized operator, spatial smoothing, local function.

[1] Research sponsored in part by the Office of Scientific Research.

be indicated briefly as follows. The theorem below does not involve any object playing the role of a physical space, but only a (real) Hilbert space $\boldsymbol{H}'$ which abstracts the real L_2 over physical space. There is assumed given in $\boldsymbol{H}'$ a positive selfadjoint operator B, which plays essentially the role of the 'single-particle hamiltonian.' The associated symmetric quantum field and its dynamics are then mathematically well-defined, and involve in particular a nonnegative, self-adjoint operator H, the 'field hamiltonian,' and an operator-valued function Φ on $\boldsymbol{H}'$, such that $\Phi(f)$ is a mathematical correlative to the formal object $\int\phi(x)f(x)dx$, where $\phi(x)$ is the "quantum field" at a fixed time, say $t=0$, and f is a given function on physical space. When V is a given selfadjoint operator affiliated with the ring of operators determined by the $\phi(f), f\in\boldsymbol{H}'$, and satisfying the indicated finite moment conditions relative to the "vacuum" expectation value functional of the field, there exists a natural selfadjoint formulation of $H+V$, which is semibounded, enjoys useful regularity and irreducibility properties, and satisfies the earlier-indicated modified Lie formula; in the event that B^{-1} is compact, has a unique one-dimensional proper subspace at the bottom of its spectrum; etc.

The proof requires of field theory only the 'real wave representation' given in [2], and uses apart from this essentially only the Lie and DuHamel formulas in the context of semigroup theory. The starting point is the formula of Mehler [3], as in the work of Nelson [4], which sketches an argument for the semiboundedness of certain operators of the form $H+V$; no use is here made of the theory of Markoff processes which is fundamental in [4].

The interaction hamiltonians V of concrete interest are typically mathematical correlatives to the formal objects $\int p(\phi(x))f(x)dx$, where p is a given real polynomial, and f is a regular function of compact support on space. Such correlatives exist as yet only in the case in which $\boldsymbol{H}'=L_2(G)$, G being a locally compact abelian group, and B is a G-invariant operator whose spectral function $B(\cdot)$ on the dual group (physically, the "energy-momentum dependence law") is reasonably large near infinity. The hypotheses of the theorem regarding V follow from the theory of renormalized products of quantum fields in this context, when $B(\cdot)$ satisfies a more stringent growth property near infinity, as in the case of a two-dimensional relativistic space-time. The generalized hyperbolicity properties of the C^*-dynamics, such as are indicated by Theorem 3 of [1] require in contrast strong restriction of B; the associated classical equation, $u''+B^2u=0$, must have similar properties (u being a function on R^1 to $\boldsymbol{H}'$); it suffices if $\sin(tB)/B$ has compact support in G depending continuously on t, as in the relativistic case $G=R^n$, $B^2=m^2I-\Delta$ (m = "mass" = constant).

On the whole, the dynamical aspects of the theory initiated in [1] (as contrasted to the essentially static aspects concerning local nonlinear functions of quantum fields) are much clarified by the present work. There remain interesting and important questions in the case of a two-dimensional space-time, notably of the structure of the vacuum and of the character of the temporal asymptotics. In higher-dimensional space-times, relativistic equations are outside the scope of the present method, but the approximation thereto in which the interaction is modified by the replacement of the field by its average over an arbitrarily small spatial region falls under Theorem 1. It is planned to treat these matters in later work.

I have received preprints of J. Glimm and A. Jaffe which follow up the direct approach of [1] in the particular case $p(\lambda)=\text{const. } \lambda^4$, in a two-dimensional relativistic space-time, by methods apparently essentially pecular to this case, and which involve a certain degree of "physical license."

3. In the formal presentation, I employ explicit mathematical terminology, without further essential colloquialisms, and the notation of [1, II], except that $\boldsymbol{H}'$ will denote an *arbitrary* real Hilbert space, and B any given selfadjoint operator in $\boldsymbol{H}'$ such that $B \geqq \epsilon I$ for some $\epsilon > 0$. In addition, $\boldsymbol{A}$ will denote the ring (of operators, in the Murray-von Neumann sense) generated by the bounded functions of the $\phi(x)$, for $x \in \boldsymbol{D}(C)$; $\boldsymbol{M}$ the collection of all normal operators in $\boldsymbol{K}$ whose spectral projections lie in $\boldsymbol{A}$; E the functional given by the equation: $E(T)=\langle Tv, v\rangle$ (defined only when $v \in \boldsymbol{D}(T)$); $L_p(\boldsymbol{A})$ the collection of all operators T in $\boldsymbol{M}$ such that the norm: $\|T\|_p = \| |T|^{p/2} v\|$ is finite. For $p \geqq 2$, the subset $L_p(\boldsymbol{A})v$ of $\boldsymbol{K}$ will be denoted as $\boldsymbol{K}_p$, as will the completion of $\boldsymbol{K}$ in the norm: $\|u\|_p = \|T\|_p$ if $u = Tv$, $T \in \boldsymbol{M}$, for $p<2$; in the topology derived from the $\|\cdot\|_p$-norm, $\boldsymbol{K}_p$ will be denoted as $[\boldsymbol{K}_p]$. (The real wave representation [2] provides an explicit isomorphism between the indicated L_p-type spaces and conventional L_p-spaces over a measure space.) The common part of all the $\boldsymbol{K}_p$ for $p<\infty$ will be denoted as $\boldsymbol{D}$; in the topology of convergence in each $\|\cdot\|_p$-norm, as $[\boldsymbol{D}]$.

THEOREM. *Let V denote an arbitrary selfadjoint operator in $\boldsymbol{K}$ such that V and e^{-V} are in $L_p(\boldsymbol{A})$ for all $p<\infty$. Then there exists a unique selfadjoint operator H' in $\boldsymbol{K}$ with the properties*:

(i) (*Formal identification*). *For $u \in \boldsymbol{D}_\infty(H)$, $H'u = Hu + Vu$.*

(ii) (*Characterization*). *If $\{f_n\}$ is any sequence of real functions on R^1 of compact support such that $f_n(\lambda) \to \lambda$ and $|f_n(\lambda)| \leqq |\lambda|$, for all $\lambda \in R^1$, then $H + f_n(V) \to H'$ in the sense that the corresponding one-parameter unitary groups converge. Conversely, $H' - f_n(V) \to H$ in* *the same sense.*

(iii) (*Regularity*). *The map* $(t, u) \to e^{-tH}u$ *carries* $[0, \infty) \times [\boldsymbol{D}]$ *continuously into* $[\boldsymbol{D}]$.

(iv) (*Parametric continuity and semiboundedness*). H' *is a continuous function of* V *relative to any set on which all* $\|V\|_n$ *and* $\|e^{-V}\|_n$ *are bounded, in the topology of convergence in every* $\|\cdot\|_n$*-norm; and* H' *is uniformly bounded from below on any such set. For* $a > 1$, $H \leqq aH' + bI$ *for suitable* b.

(v) (*Irreducibility*). *There are no nontrivial invariant subspaces under* $\boldsymbol{A}$ *and any* e^{-tH}, $t > 0$.

The salient points of the proof are as follows. From Mehler's formula for Hermite functions [3] and Hölder's inequality, it results that if A denotes the selfadjoint operator in the Hilbert space $\boldsymbol{N} = L_2(R^1, g)$, where $dg = (2\pi)^{-1/2}e^{-x^2/2}dx$, which multiplies the nth Hermite function by n, e^{-tA} is a contraction on $L_p(R^1, g)$ for all $p \in [1, \infty]$, and is a contraction from $L_2(R^1, g)$ to $L_p(R^1, g)$, for given $p < \infty$, for sufficiently large t, say $t \geqq t_0(p)$.

Applying the fact that if T is an integral operator with nonnegative kernel which is a contraction from $L_r(M)$ to $L_s(M)$ for some measure space M, then for any Banach space $\boldsymbol{B}$ the induced map T' from $L_r(M, \boldsymbol{B})$ to $L_s(M, \boldsymbol{B})$ is also a contraction, it follows that the n-fold tensor product of e^{-tA} with itself is a contraction from $L_2(R^n, g^n)$ to $L_p(R^n, g^n)$ for $t > t_0$ (p) being independent of n). Employing the real wave representation [2], it follows that if T is any self-adjoint operator on $\boldsymbol{H}$ having pure point spectrum, and such that $T \geqq mI$ $(m > 0)$, then $\exp[-td\Gamma(T)]$ is a contraction from $\boldsymbol{K}$ to $[\boldsymbol{K}_p]$, if $t > m^{-1}t_0(p)$. By an approximation argument and Fatou's lemma, the pure point spectrum assumption may be removed. Moreover, the same operator is a contraction from $[\boldsymbol{K}_r]$ to $[\boldsymbol{K}_s]$ for any $r \geqq s$.

Now let $\boldsymbol{C}$ denote the collection of all operators V satisfying the hypothesis of the theorem, and such that $\|V\|_n$ and $\|e^{-V}\|_n$ have fixed (but arbitrary) bounds; let $\boldsymbol{C}_b$ denote the subset consisting of those V which are bounded. For V in $\boldsymbol{C}_b$, the DuHamel and Lie formulas are applicable:

$$e^{-t(H+V)} = e^{-tH} + \int_0^t e^{-(t-s)H}Ve^{-s(H+V)}ds = \lim_n (e^{-tV/n}e^{-tH/n})^n.$$

Now note the general fact that if A and B are selfadjoint operators in Hilbert space, if B is bounded, and if $e^{-A/2}e^{-B}e^{-A/2} \leqq cI$, then also $e^{-(A+B)} \leqq cI$; this follows from the monotonicity of the square root operation on positive selfadjoint operators together with the cited Lie formula. If in particular, for $s = t_0(p)$ for any $p > 2$, $A = 2sH$ and

$B=sV$ with $V\epsilon C_b$, then $e^{-A/2}$ is a contraction from K to $[K_p]$; by Hölder's inequality, e^{-B} is bounded from $[K_p]$ to K, its operator bound being bounded on C_b; and $e^{-A/2}$ is a contraction from K to K; and it follows that $e^{-s(H+V)}$ is uniformly bounded for $V\in C_b$. Hence, $e^{-t(H+T)}$ is uniformly bounded for $t\in[0,1]$ and $V\in C_b$. By a similar argument, it is uniformly bounded in norm as an operator from $[K_r]$ to $[K_s]$ for any fixed r and s, $r>s$.

By the DuHamel formula, for arbitrary V and V' in C_b, and $u\in K$,

$$e^{-t(H+V)}u - e^{-t(H+V')} = \int_0^t e^{-(t-s)(H+V')}(V - V')e^{-s(H+V)}uds.$$

Applying the bounds just derived, it follows that for $u\in D$ and V arbitrary in C, $e^{-t(H+f_m(V))}u$ is convergent as $m\to\infty$, say to $S_0(t)u$, where $\|S_0(t)\|$ is bounded for t in I. It follows in turn that $S_0(t)$ may be uniquely extended to a bounded operator $S(t)$ on all of K; and that $S(\cdot)$ is a continuous one-parameter group of selfadjoint operators in K. It has therefore the form $S(t)=e^{-tH'}$ for a unique semibounded selfadjoint operator H' in K.

It is not difficult to verify (e.g. by using the Laplace transform and the Stone-Weierstrass theorem) that if the T_n and T are uniformly semibounded selfadjoint operators in a Hilbert space such that $e^{-tT_n}\to e^{-tT}$ for $t>0$, then $e^{itT_n}\to e^{itT}$ for all $t\in R^1$. This implies the first part of (ii). It follows from the bounds earlier obtained that $e^{-tH'}$ is bounded from $[K_r]$ to $[K_s]$, whenever $r>s$, uniformly when V varies over C. In particular, $e^{-tH'}$ leaves D invariant, and acts continuously on $[D]$; the continuity of $e^{-tH'}u$ as a function of t, for $u\in[D]$, follows by regularization. Applying the DuHamel formula to $e^{-t(H'-V_1)}-e^{-t(H'-V_2)}$, where V_1 and V_2 are in C_b the remainder of (ii) follows, and a further use of the formula establishes (i). A similar argument establishes parametric continuity. The uniform semiboundedness of H' as a function of V follows from the construction. Using the semiboundedness with V replaced by cV, with c a constant, the indicated domination of H by H' follows.

If U is any unitary operator on K which commutes with both $e^{-tH'}$, for some fixed t, and every element of A, then by the second part of (ii) it commutes also with H; but by the uniqueness of the vacuum [5], only scalars commute with both H and every element of A.

Typical of the further developments emergent from the theorem are

COROLLARY 1 (EXISTENCE OF NORMALIZABLE VACUUM). *If B^{-1} is compact, there exists a unique vector v' in K such that:*

(a) *$H'v'=\lambda v'$ for some λ;*

(b) *$\langle H'u, u\rangle$ has a positive lower bound as u varies over the elements* *of* $D(H')$ *which are orthogonal to* v' *and have unit norm;*

(c) $\|v'\|=1$ *and* $v'=Av$, *with* $A\in M$, $A\geqq 0$, *and with trivial null-space.*

The compactness of B^{-1} implies that of $(I+H)^{-1}$; inversion of the inequality in (iv) shows that $e^{-tH'}$ is also compact. Noting (v), the corollary follows directly from [6].

COROLLARY 2. *The conclusion of Theorem* 3 *of* [1] *holds (unprovisionally) with* $H(f)$ *replaced by* $H'(V(f))$, *for* $V(f)=\int :p(\phi(x)):f(x)dx$, *if* p *is any real polynomial which is bounded from below,* f *being nonnegative.*

That $E(V^{2n})$ is finite is shown in [7], so that it is only necessary to show that e^{-nV} has finite expectation value. Adapting Nelson's observation thereto in a special case [4], it suffices to show that if $g_x(y)=g(x+y)$ and $Z=\int :p(\phi(g_x)):f(x)dx-\int :p(\phi(x)):f(x)dx$, where $\hat{g}(k)=1$ if $|k|<\lambda$ and 0 otherwise, and if h is the function on R^1 given by: $h(x)=1$ if $|x|\geqq 1$, $h(x)=0$ for $|x|<1$, then $E(h(Z))=O(\lambda^{-a})$ as $\lambda\to\infty$, for some $a>0$. But $E(h(Z))\leqq E(Z^2)$, so that by Schwarz' inequality it suffices to show that $E(Z^2)=O(\lambda^{-a})$ when $p(\lambda)=\lambda^n$. An explicit computation of $E(Z^2)$ and estimation by the Hausdorff-Young inequality similar to that given in [7], shows that $E(Z^2)=O(\|B(\cdot)^{-1}-B(\cdot)^{-1}\hat{g}\|_p)$ for suitable $p>1$, and this has the required order when $B(k)=(m^2+k^2)^{1/2}$.

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