NORMS ON QUOTIENT SPACES

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Communicated by Bertram Yood, June 23, 1967

1. Perturbation classes. Let $\mathcal{S}$ be a subset of a Banach space $\mathfrak{A}$ over the complex numbers, and assume that $\alpha\mathcal{S}\subset\mathcal{S}$ for each scalar $\alpha\neq 0$. Let $P(\mathcal{S})$ denote the set of elements of $\mathfrak{A}$ that perturb S into itself, i.e., $P(\mathcal{S})=\{a\in\mathfrak{A}\colon a+s\in\mathcal{S} \text{ for all } s\in\mathcal{S}\}$.

PROPOSITION 1.1. *$P(\mathcal{S})$ is a linear subspace of $\mathfrak{A}$. If $\mathcal{S}$ is an open subset of $\mathfrak{A}$, then $P(\mathcal{S})$ is closed.*

PROPOSITION 1.2. *Let $\mathcal{S}_1\subset\mathcal{S}_2$ be two such subsets, and assume that $\mathcal{S}_1$ is open and $\mathcal{S}_2$ does not contain any boundary point of $\mathcal{S}_1$. Then $P(\mathcal{S}_2)\subset P(\mathcal{S}_1)$.*

PROPOSITION 1.3. *Assume that $\mathfrak{A}$ is a Banach algebra with identity e. Let G denote the set of invertible elements in $\mathfrak{A}$. If $G\mathcal{S}\subset\mathcal{S}$, then $P(\mathcal{S})$ is a left ideal. If $\mathcal{S}G\subset\mathcal{S}$, then $P(\mathcal{S})$ is a right ideal.*

PROPOSITION 1.4. *$P(G)=R$, the radical of $\mathfrak{A}$.*

Let G_l (G_r) denote the set of left (right) invertible elements of $\mathfrak{A}$, and let H_l (H_r) denote the set of elements of $\mathfrak{A}$ that are not left (right) topological divisors of zero.

THEOREM 1.5. $P(H_l)\subset P(G_l)=R=P(G_r)\supset P(H_r)$.

Let X be a Banach space, and let $B(X)$ $[\mathcal{K}(X)]$ denote the set of bounded (compact) linear operators on X. Take $\mathfrak{A}=B(X)/\mathcal{K}(X)$ and let π be the canonical homomorphism from $B(X)$ to $\mathfrak{A}$. Set

$$\Phi(X)=\pi^{-1}(G),\qquad \Phi_l(X)=\pi^{-1}(G_l),\qquad \Phi_r(X)=\pi^{-1}(G_r).$$

It is well known [6] that $\Phi_l(X)$ consists of those operators having finite nullity and closed, complemented ranges, and that $\Phi_r(X)$ consists of those operators having complemented null spaces and closed ranges with finite codimensions. $\Phi(X)=\Phi_l(X)\cap\Phi_r(X)$ is the set of Fredholm operators on X.

THEOREM 1.6. $P(\Phi)=P(\Phi_l)=P(\Phi_r)=\pi^{-1}(R)$.

Let Z be any subset of $\{0,\pm 1,\pm 2,\cdots,\pm\infty\}$, and let Φ_z be the collection of those operators $A\in\Phi_l(X)\cup\Phi_r(X)$ such that $i(A)\in Z$, where $i(A)=\dim N(A)-\dim N(A')$.

THEOREM 1.7. $P(\Phi_z)=\pi^{-1}(R)$.

2. **Measures of noncompactness.** Let X, Y be Banach spaces, and denote the set of bounded (compact) linear operators from X to Y by $B(X, Y)$ $[\mathcal{K}(X, Y)]$. Let S_X denote the unit ball in X. For any bounded subset Ω of X let $q(\Omega)$ denote the greatest lower bound of the numbers r such that Ω can be covered by a finite collection of spheres of radius r. For $A \in B(X, Y)$ set $\|A\|_q = q[A(S_X)]$. Let $\|A\|_m$ denote the greatest lower bound of all numbers η such that $\|Ax\| \leqq \eta\|x\|$ for all x in some subspace having finite codimension. Let π denote the canonical homomorphism of $B(X, Y)$ into $B(X, Y)/\mathcal{K}(X, Y)$.

PROPOSITION 2.1. *Both* $\|\cdot\|_q$ *and* $\|\cdot\|_m$ *are seminorms and satisfy* $\|BA\|_q \leqq \|B\|_q\|A\|_q$, $\|BA\|_m \leqq \|B\|_m\|A\|_m$, $\|A\|_q \leqq \|\pi(A)\|$, $\|A\|_m \leqq \|\pi(A)\|$, $\|A+K\|_q = \|A\|_q$, $\|A + K\|_m = \|A\|_m$ *for* $K \in \mathcal{K}(X, Y)$.

THEOREM 2.2. $\|A\|_q/2 \leqq \|A\|_m \leqq 2\|A\|_q$.

DEFINITION 2.3. A Banach space X will be said to have the *compact approximation property with constant* γ if for each $\epsilon > 0$ and finite set of points $x_1, \cdots, x_n$ in X there is an operator $K \in \mathcal{K}(X)$ such that $\|I-K\| \leqq \gamma$ and $\|x_j - Kx_j\| < \epsilon$ for $1 \leqq j \leqq n$.

THEOREM 2.4. *If* Y *has the compact approximation property with constant* γ, *then* $\|\pi(A)\| \leqq \gamma\|A\|_q$. *Thus* $B(X, Y)/\mathcal{K}(X, Y)$ *is complete with respect to the norms induced by* $\|\cdot\|_q$ *and* $\|\cdot\|_m$.

3. **Semi-Fredholm operators.** An operator $A \in B(X, Y)$ is in $\Phi_+(X, Y)$ if it has finite nullity and closed range.

THEOREM 3.1. *An operator* A *is in* $\Phi_+(X, Y)$ *if and only if for each Banach space* Z *there is a constant* C *such that* $\|T\|_m \leqq C\|AT\|_m$, $T \in B(Z, X)$. *The constant does not depend on* Z.

COROLLARY 3.2. *If* $A \in \Phi_+(X, Y)$ *and* X *has the compact approximation property, then* $\|\pi(T)\| \leqq C\|\pi(AT)\|$, $T \in B(Z, X)$, *for any Banach space* Z.

DEFINITION 3.3. For $A \in B(X, Y)$ set $q_A = \text{glb } q[A(\Omega)]/q(\Omega)$, where the glb is taken over all bounded subsets Ω of X.

THEOREM 3.4. $A \in \Phi_+(X, Y)$ *if and only if* $q_A \neq 0$.

An operator $A \in B(X, Y)$ is in $\Phi(X, Y)$ if its range is closed and has finite codimension.

THEOREM 3.5. $A \in \Phi_-(X, Y)$ *if and only if* $\beta(A-K) < \infty$ *for all* $K \in \mathcal{K}(X, Y)$, *where* $\beta(E) = \text{codim } \overline{R(E)}$.

THEOREM 3.6. *$A\in\Phi_-(X, Y)$ if and only if for each Z there is a constant C such that $\|T\|_m \leqq C\|TA\|_m$, $T\in B(Y, Z)$. The constant C is independent of Z.*

We now consider the case $X=Y$. Let $r_\sigma(A)$ denote the spectral radius of an operator A.

THEOREM 3.7. *If $\|A^n\|_m<1$ for some $n\geqq 1$, then $I-A\in\Phi(X)$ and $i(I-A)=0$.*

THEOREM 3.8.

$$r_\sigma[\pi(A)] = \lim_{n\to\infty} \|A_n\|_m^{1/n} = \lim_{n\to\infty} \|A^n\|_q^{1/n} = \max_{\lambda\in\sigma_e(A)} |\lambda|,$$

where $\sigma_e(A)$ denotes the essential spectrum of A according to any of the usual definitions [8], [9].

COROLLARY 3.9. *$r_\sigma[\pi(A)]\geqq q_A$. Hence an operator in $\Phi_+(X)$ cannot be a Riesz operator.*

DEFINITION 3.10. A space X has the *range* property if for each $\epsilon>0$ and each $A\in B(X)$ with $\dim N(A)=\infty$ there is a $T\in B(X)$ such that $\|T\|_q=1$ and $q[T(S_X)\backslash N(A)]<\epsilon$. All subprojective [10] spaces have the range property.

THEOREM 3.11. *If X has the range property, then $A\in\Phi_+(X)$ if and only if $\|T\|_q\leqq C\|AT\|_q$ for all $T\in B(X)$.*

THEOREM 3.12. *If X is subprojective and $\pi(A)$ is not a left zero divisor then $A\in\Phi_+(X)$.*

COROLLARY 3.13. *If X is subprojective and has the compact approximation property, then every topological left zero divisor in $B(X)/\mathcal{K}(X)$ is a left zero divisor.*

THEOREM 3.14. *If X is superprojective* [10] *and $\pi(A)$ is not a right zero divisor, then $A\in\Phi(X)$.*

COROLLARY 3.15. *If X is both subprojective and superprojective, then every element of $B(X)/\mathcal{K}(X)$ which is not a zero divisor is invertible.*

4. **Remarks.** Some of the results of §1 were also obtained by B. Gramsch [12]. The q-seminorm was studied by Gol′denšteĭn, Gokhberg, Markus [1], [2] and Darbo [3]. The basic idea goes back to Kuratowski [11]. For the q-seminorm Propostion 2.1 was proved in [1]. The compact approximation property is weaker than the metric approximation property of Grothendieck [4] and is similar to one of Bonsall [5].

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