A UNIVERSAL MODEL FOR DYNAMICAL SYSTEMS WITH QUASI-DISCRETE SPECTRUM

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Let K_d denote the circle group (unit interval with addition mod 1) with the discrete topology, let $\hat{K}_d$ be its dual (the Bohr compactification of the integers), and let $\hat{K}_d^\omega$ denote the direct product of countably many copies of $\hat{K}_d$. We define an affine transformation $\hat{\phi}$ of the group $\hat{K}_d^\omega$ by

$$\hat{\phi}(x_1, x_2, x_3, \cdots) = (x_1 + \hat{a}, x_1 + x_2, x_2 + x_3, \cdots),$$

where $\hat{a} \in \hat{K}_d$ is defined by

$$\hat{a}(t) = t \qquad (t \in K_d).$$

Implicit in the proof of the representation theorem of Halmos and von Neumann [5] for measure-preserving transformations with discrete spectrum is the fact that every such system is a factor (homomorphic image) of the translation $x \to x + \hat{a}$ on $\hat{K}_d$. Moreover, it follows easily from the fact that any subgroup Γ of K_d is the union of its finitely generated subgroups (as is any group) that any system with discrete spectrum is an inverse limit of direct products of ergodic translations on the circle group K (usual topology) and cyclic permutations of finite sets. A discussion of such inverse limits is given in [2]. In the present note we extend these two results to the ergodic dynamical systems Φ with quasi-discrete spectrum that were introduced by L. M. Abramov [1] in the measure-theoretic case and by F. Hahn and W. Parry [3] in the topological case.

In each case, by virtue of the respective representation theorems, we can assume without loss of generality that the system $\Phi = (G, \mathfrak{B}, \mu, \phi)$ is such that G is a compact abelian group, $\mathfrak{B}$ is the class of Baire subsets of G, μ is normalized Haar measure, and ϕ is an ergodic, continuous, affine transformation of G with quasi-discrete spectrum [1], [3], [4].

By an *affine* transformation of G we mean a map $\phi: G \to G$ of the form $\phi(x) = \tau(x) + a$, where τ is an automorphism of G and $a \in G$. Let us denote by σ the endomorphism of G defined by $\sigma(x) = \tau(x) - x$. Then ϕ is said to have *quasi-discrete spectrum* provided that

$$\bigcap_{n=1}^{\infty} \sigma^n G = (0).$$

THEOREM 1. *Every ergodic dynamical system with quasi-discrete spectrum is a factor of the system* $\hat{\Phi} = (\hat{K}_d^\omega, \hat{\mathfrak{B}}, \hat{\mu}, \hat{\phi})$, *where* $\hat{\mu}$ *is the Haar measure on the class* $\hat{\mathfrak{B}}$ *of Baire subsets of* $\hat{K}_d^\omega$.

REMARK. This theorem, as well as Theorem 2 below, can be read in any one of three categories—affine transformations on compact abelian groups with continuous homomorphisms for the maps, classical dynamical systems with continuous maps, or abstract (measure-theoretic) systems with measure-preserving maps. In each case the theorem asserts the existence of a map ψ of $\hat{K}_d^\omega$ onto G such that $\psi\hat{\phi} = \phi\psi$. We indicate this by $\psi \colon \hat{\Phi} \to \Phi$. We prove these theorems in the first category indicated, their validity in the other two following from the representation theorems of [1] and [3].

PROOF. We define the homomorphism ψ by describing its adjoint $\psi^* \colon \Gamma \to K_d^\omega$, where K_d^ω, the direct sum of countably many copies of K_d, is the dual of $\hat{K}_d^\omega$, and $\Gamma = \hat{G}$ is the dual of G. Specifically, we set

(1) $$\psi^*(\gamma) = (\gamma(a), \gamma(\sigma a), \gamma(\sigma^2 a), \cdots).$$

We need to show that ψ^* is a monomorphism, and that $\psi\hat{\phi} = \phi\psi$. The latter is equivalent, after observing that $\psi(\hat{a}, 0, 0, \cdots) = a$, to the condition $\psi\hat{\sigma} = \sigma\psi$, where

$$\hat{\sigma}(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, \cdots).$$

This condition, in turn, is equivalent to $\hat{\sigma}^*\psi^* = \psi^*\sigma^*$, which follows easily from (1). Since ψ^* is readily seen to be a homomorphism, the proof reduces to showing that it is one-to-one.

Let us denote by Γ_n the annihilator of $\sigma^n G$. Then Γ_n is the kernel of σ^{*n} and $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$. Suppose $\gamma \in \Gamma_{n+1}$ and $\psi^*(\gamma) = 0$, i.e. $\gamma(\sigma^k a) = 0$ for each $k = 1, 2, \cdots, n$ and $\gamma(\sigma^k x) = 0$ for all $x \in G$ and $k > n$. Let $g = \sigma^{*n}\gamma$. Then

$$g(\phi x) = \gamma(\sigma^n(x + \sigma x + a))$$
$$= \gamma(\sigma^n x) + \gamma(\sigma^{n+1} x) + \gamma(\sigma^n a)$$
$$= g(x).$$

By ergodicity it follows that g is a constant, necessarily 0. Thus $\gamma \in \Gamma_n$. By a repetition of this argument, it follows that $\gamma = 0$, i.e. the kernel of ψ^* is trivial. ∎

Note that the mapping ψ^* is an embedding of the quasi-eigenvalue system of Φ (as defined in [3]) into that of $\hat{\Phi}$. Let us call a group $H \subseteq K_d^\omega$ *shift-invariant* if $\hat{\sigma}^* H \subseteq H$.

COROLLARY. *If* Φ *is any ergodic dynamical system, then its system*

*of quasi-eigenvalues is isomorphic to a shift-invariant subgroup of K_d^ω.
Conversely, any such subgroup is the quasi-eigenvalue system of some
ergodic dynamical system with quasi-discrete spectrum.*

Suppose that $\gamma^1, \gamma^2, \cdots, \gamma^k \in K_d^\omega$, and let Γ be the smallest shift-invariant group containing them. Then $\Gamma \subseteq K_d^n$, the subgroup of elements $\gamma \in K_d^\omega$ with at most the first n components different from 0, for some n. Moreover, $\Gamma \subseteq \Gamma^1 \oplus \Gamma^1 \oplus \cdots \oplus \Gamma^1 \oplus (0)$ (n factors) for some finitely generated subgroup Γ^1 of K_d. (Let Γ^1 be generated by γ_i^j, $i = 1, \cdots, n$; $j = 1, \cdots, k$.) Since Γ^1 is a direct sum of cyclic groups, we obtain a decomposition of the above sum into the quasi-eigenvalue systems of some special dynamical systems to be described now.

We shall call the dynamical system $\Phi_\alpha = (G_\alpha, \mathfrak{B}_\alpha, \mu_\alpha, \phi_\alpha)$ *special* if μ_α is Haar measure on the Baire sets $\mathfrak{B}_\alpha$ of G_α, where either $\alpha \in K$ and $G_\alpha = K^n$ or $\alpha \in Z_t$ (the additive group of integers mod t) and $G_\alpha = Z_t^n$, and if

$$\phi_\alpha(x_1, \cdots, x_n) = (x_1 + \alpha, x_1 + x_2, \cdots, x_{n-1} + x_n).$$

In the first case, it is assumed that α is irrational, and in the second that the order of α is t. Thus Φ_α is ergodic. Its system of quasi-eigenvalues is $(\alpha) \oplus (\alpha) \oplus \cdots \oplus (\alpha) \oplus (0)$. Such systems were considered in [3] under the name of "diagonal" transformations.

THEOREM 2. *$\hat{\Phi}$ is an inverse limit of products of special systems. More generally, any dynamical system with quasi-discrete spectrum is an inverse limit of factors of products of special systems.*

Clearly, those familiar properties (see [1], [3], [4]) of dynamical systems with quasi-discrete spectrum, such as zero entropy, that are preserved under the formation of products, factors and inverse limits follow readily from Theorems 1 and 2.

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