# A STURM THEOREM FOR STRONGLY ELLIPTIC SYSTEMS AND APPLICATIONS

BY KURT KREITH

Communicated by M. H. Protter, April 30, 1969

A recent announcement in this Bulletin [1] deals with Sturm comparison theorems for elliptic equations and systems of ordinary differential equations based on a generalization of the Picone identity [2, p. 228]: if $v(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{u}{v}(au'v - guv')\right] = u(au')' - \frac{u^2}{v}(gv')' + (a-g)u'^2 + g\left(u' - \frac{u}{v}v'\right)^2. \tag{1}$$

The purpose of this note is to present a generalization of (1) to strongly elliptic systems and to describe some of its consequences.

Let $A_{ij}(x)$, $G_{ij}(x)$, $C(x)$, and $H(x)$ denote real symmetric $n \times n$ matrices whose components are defined in a smooth, bounded closed domain $\bar{D}$ in $R_m$ and which satisfy $G_{ij} = G_{ji}$, $A_{ij} = A_{ji}$ for $i, j = 1, \cdots, m$. The components of $A_{ij}$ and $G_{ij}$ are to be of class $C^2$ in $\bar{D}$ while the components of $C$ and $H$ are continuous. Let $V(x)$ be a $n \times n$ matrix of class $C^2$ which is nonsingular in $\bar{D}$ and is "prepared" in the sense that it satisfies

$$V^* \sum_{j=1}^{m} G_{ij} \frac{\partial V}{\partial x_j} \quad \text{is symmetric for } i = 1, \cdots, m, \tag{2}$$

and let $U(x)$ be a $n \times 1$ matrix of class $C^2$. Then (1) has the following generalization:

$$\begin{aligned}\sum_i \frac{\partial}{\partial x_i}\left[U^* \sum_j A_{ij} \frac{\partial U}{\partial x_j} - U^* \sum_j G_{ij} \frac{\partial V}{\partial x_j} V^{-1} U\right] &= U^* \sum_{i,j} \frac{\partial}{\partial x_i}\left(A_{ij} \frac{\partial U}{\partial x_j}\right) - U^* \sum_{i,j} \frac{\partial}{\partial x_i}\left(G_{ij} \frac{\partial V}{\partial x_j}\right) V^{-1} U \\ &+ \sum_{i,j} \frac{\partial U^*}{\partial x_i}(A_{ij} - G_{ij}) \frac{\partial U}{\partial x_j} \\ &+ \sum_{i,j}\left[\frac{\partial U}{\partial x_i} - \frac{\partial V}{\partial x_i} V^{-1} U\right]^* G_{ij}\left[\frac{\partial U}{\partial x_j} - \frac{\partial V}{\partial x_j} V^{-1} U\right].\end{aligned} \tag{3}$$

Let $K$ and $L$ be strongly elliptic operators defined by

$$K[U] \equiv \sum_{i,j} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial U}{\partial x_j}\right) + CU$$

and

$$L[V] \equiv \sum_{i,j} \frac{\partial}{\partial x_i}\left(G_{ij}\frac{\partial V}{\partial x_j}\right) + HV,$$

respectively. Then (3) leads to the following comparison theorem, generalizing related results of Kuks [3] and Bochenek [4].

THEOREM 1. *Let $U(x)$ be a nontrivial solution of*

$$U^*K[U] \geqq 0 \qquad \text{in } D$$

$$\sum_{i,j} A_{ij}\frac{\partial U}{\partial x_j}\cos(\nu, x_i) + SU = 0 \qquad \text{on } \partial D - \Gamma,$$

$$U = 0 \qquad \text{on } \Gamma \subset \partial D,$$

*where $S(x)$ is a symmetric matrix continuous on $\partial D - \Gamma$ and $\nu$ denotes the exterior normal to $\partial D$. Let $V(x)$ be a solution of $L[V]=0$ in $D$ satisfying* (2). *If*

(i) $\int_D \sum_{i,j} \frac{\partial U^*}{\partial x_i}(A_{ij}-G_{ij})\frac{\partial U}{\partial x_j}\,dx \geqq 0,$

(ii) $\int_D U^*(H-C)U\,dx \geqq 0,$

(iii) $S - \frac{\partial V}{\partial \nu}V^{-1} \geqq 0$ *on* $\partial D - \Gamma$,

*then* det $V$ *has a zero in* $D \cup \Gamma$.

REMARK. All inequalities are to be interpreted as indicating that certain matrices are semidefinite.

As an application of this comparison theorem we cite the following "maximum principle" for strongly elliptic systems.

THEOREM 2. *Let $U(x)$ satisfy $U^*K[U] \geqq 0$ in a domain $D$ where $C(x) \leqq 0$. Then there is no smooth proper subdomain $D' \subset D$ such that*

$$\sum_{i,j} A_{ij}\frac{\partial U}{\partial x_j}\cos(\nu, x_i) + SU = 0 \qquad \text{on } \partial D' \cap D,$$

$$\sum_{i,j} A_{ij}\frac{\partial U}{\partial x_j}(\cos \nu, x_i) = 0 \qquad \text{on } \partial D' \cap \partial D,$$

*with $S(x) > 0$ on $\partial D' \cap D$.*

In the special case where the components of $K[U]$ are given by

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u_k}{\partial x_j}\right) + \sum_l c_{kl}u_l; \qquad k = 1, \cdots, n,$$

Theorem 2 implies the following.

COROLLARY 1. *Let $U(x)$ satisfy $U^* K[U] \geqq 0$ in a domain $D$ where $C(x) \leqq 0$. If there exists an $x_0 \in D$ such that all the $u_k(x)$ have a positive maximum or a negative minimum at $x_0$, then $U(x)$ is constant in $D$.*

Finally we note a lower bound for the first eigenvalue $\lambda_1$ of

$$K[U] + \lambda PU = 0 \qquad \text{in } D,$$

$$\sum A_{ij}\frac{\partial U}{\partial x_j}\cos(\nu, x_i) + SU = 0 \qquad \text{on } \partial D - \Gamma$$

$$U = 0 \qquad \text{on } \Gamma,$$

where $P$ is real, positive definite, and symmetric in $D$.

THEOREM 3. *Let $V(x)$ be a nonsingular $n \times n$ matrix class $C^2$ for which $A_{ij}(\partial V/\partial x_j)V^{-1}$ is symmetric for $i, j = 1, \cdots, m$. If $V$ satisfies*

$$\sum A_{ij}\frac{\partial V}{\partial x_j}\cos(\nu, x_i) + TV = 0 \qquad \text{on } \partial D - \Delta,$$

$$V = 0 \qquad \text{on } \Delta,$$

*where $\Delta \subset \Gamma$ and $S \geqq T$ on $\partial D - \Gamma$, then*

$$\lambda_1 I \geqq \inf_{x\in D}(-P^{-1}K[V]V^{-1}).$$

Proofs of these results and other applications of (3) will be presented elsewhere.

## REFERENCES

1. J. B. Diaz and J. R. McLaughlin, *Sturm comparison theorems for ordinary and partial differential equations*, Bull. Amer. Math. Soc. **75** (1969), 335–339.

2. E. L. Ince, *Ordinary differential equations*, Dover, New York, 1956.

3. L. M. Kuks, *Sturm's theorem and oscillation of solutions of strongly elliptic systems*, Dokl. Akad. Nauk. SSSR **142** (1962), 32–35=Soviet Math. Dokl. **3** (1962), 24–27.

4. J. Bochenek, *On eigenvalues and eigenfunctions of strongly elliptic systems of differential equations of second order*, Prace Mat. **12** (1968), 171–182.

UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616