SMOOTH HOMOTOPY PROJECTIVE SPACES

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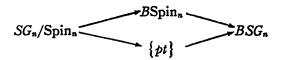
Introduction. In [5] we considered certain fixed point free involutions on Brieskorn manifolds as weakly complex bordism elements. In [4] we considered associated examples of smooth normal invariants for real projective spaces, settling the realizability question for dimensions $\not\equiv 1 \mod 4$ and the desuspendability question for dimensions 4k+1. The object of this study is the classification of these smooth normal invariants given by the Brieskorn examples. Our results overlap somewhat with Atiyah and Bott [2] as well as Browder [3], but our methods are entirely different and our results rather more refined. Full details of these and related results will appear elsewhere.

1. Smooth normal invariants. Following Sullivan [6], we regard a smooth normal invariant of a space X as an element of [X, G/O]. Of course, we have $G/O \cong SG/SO$. We need the fibers SG/Spin of $BSpin \rightarrow BSG$ and $SO/Spin \simeq P^{\infty}$ of $BSpin \rightarrow BSO$. The spaces SG/SO, SG/Spin, SO/Spin have their Whitney H-space structures under which the sequence

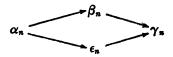
$$SO/Spin \rightarrow SG/Spin \rightarrow SG/SO$$

is a multiplicative fibration.

A map μ : $SG/\operatorname{Spin} \to BO$ is constructed as follows. Let γ_n denote the universal fiber space over BSG_n with fiber S^{n-1} , β_n the pullback to $B\operatorname{Spin}_n$, and α_n the pullback to $SG_n/\operatorname{Spin}_n$; also, let ϵ_n denote the S^{n-1} fibration over a point. Corresponding to the commutative diagram



of spaces, there is the commutative diagram



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of induced S^{n-1} fibrations. Passing to Thom spaces we obtain a commutative diagram of spectra

$$N(SG/Spin)$$
 MSG

where **S** is the sphere spectrum and $N(SG/\operatorname{Spin})$ is the spectrum with $N(SG/\operatorname{Spin})_n = T(\alpha_n) \simeq S^n \wedge (SG_n/\operatorname{Spin}_n)^+$. There is a map of spectra $M\operatorname{Spin} \to bO$, where bO is the Ω -spectrum with $bO_0 = \mathbf{Z} \times BO$, defining the kO-orientation of Spin cobordism. Now the composition

$$N(SG/\mathrm{Spin}) \to M\mathrm{Spin} \to bO$$

of maps of ring spectra defines μ : $SG/Spin \rightarrow BO$ in the usual way, since N(SG/Spin) and $S \land (SG/Spin)^+$ are equivalent.

(1.1) THEOREM. The map μ : $SG/Spin \rightarrow BO$ is an H-map from the Whitney structure of SG/Spin to the tensor product structure of BO.

The following is a central fact in our study.

(1.2) THEOREM. The composition

$$P^{\infty} \simeq SO/\operatorname{Spin} \to SG/\operatorname{Spin} \to BO$$

classifies the canonical line bundle η over P^{∞} .

Since this map $\eta: P^{\infty} \to BO$ splits the map $w_1: BO \to K(Z_2, 1) \simeq P^{\infty}$, we get the following easily.

(1.3) COROLLARY. The above maps fit into a commutative diagram

$$P^{\infty} \to SG/\operatorname{Spin} \to G/O$$

$$\parallel \qquad \qquad \downarrow \mu \qquad \qquad \downarrow \nu$$

$$P^{\infty} \xrightarrow{\eta} BO \longrightarrow BSO$$

of maps of H-spaces (BO and BSO with the tensor product structure), each row being a multiplicative fibration.

Using Poincaré duality for Spin cobordism theory, we are able to make explicit computations of the kO-orientation $\mu: SG/Spin \rightarrow BO$. This amounts to interpreting $N(SG/Spin) \rightarrow MSpin$ in terms of Spin bordism of Spin manifolds.

For a finite CW complex X, we take a homotopy equivalent compact Spin manifold M^m . An element of $[M^m, SG/Spin]$ is represented by a spherical fiber bundle over M^m with Spin structural group and an SG (i.e. degree +1 fiber homotopy) trivialization. Using transverse

regularity on the SG trivialization, we obtain an element $[V^m, \partial V^m; e] \in \Omega_m^{\mathrm{Spin}}(M^m, \partial M^m)$ of degree +1 (e is the bundle projection). The Poincaré duality isomorphism sends this to an element of $\Omega_{\mathrm{Spin}}^0(M^m)$ with augmentation +1. Now applying the kO-orientation of Spin cobordism, we obtain a virtual line bundle over M^m , i.e. an element of $[M^m, BO]$. This describes the natural map $[X, \mu]: [X, SG/\mathrm{Spin}] \to [X, BO]$.

- 2. Spin cobordism of projective spaces. At this point the Brieskorn examples are brought into play. The degree q maps $h: Q_q^{4k-1} \rightarrow P^{4k-1}$ of [5, §3] may be taken to represent elements $[Q_q^{4k-1}, h] \in \Omega_{4k-1}^{\text{Spin}}(P^{4k-1})$ in a canonical way so that the following analogue of [5, Theorem 3.4] holds.
- (2.1) Theorem. The Poincaré duals $b_q^{4k-1} \in \Omega^0_{\mathrm{Spin}}(P^{4k-1})$ of the elements $[Q_q^{4k-1}, h] \in \Omega^{\mathrm{Spin}}_{4k-1}(P^{4k-1})$ satisfy $b_q^{4k-1} = q \cdot 1$.

It happens that $h: Q_1^{4k-1} \to P^{4k-1}$ is a diffeomorphism, making it clear what Spin structures to use. Actually, the above holds equally well for Sp and SU cobordism of P^{4k-1} .

To see what happens for other dimensions, just cut the maps $h\colon Q_q^{4k-1} \to P^{4k-1}$ down to a suitably small smooth regular neighborhood of $P^n \subset P^{4k-1}$, obtaining $h\colon V_q^{4k-1,n} \to M^{4k-1,n}$, say. Note that the Brieskorn examples Q_q^n of [4] are the transverse regular inverse images in Q_q^{4k-1} of $P^n \subset P^{4k-1}$. The resulting elements $b_q^n \in \Omega^0_{\mathrm{Spin}}(P^n)$ are just the inclusion induced pullbacks of the elements b_q^{4k-1} for $n \leq 4k-1$.

(2.2) COROLLARY. In $\Omega^0_{\mathrm{Spin}}(P^n)$ we have $b_q^n = q \cdot 1$.

The double covering $e: S^{4k-1} \rightarrow P^{4k-1}$ defines the element $[S^{4k-1}, e] \in \Omega^{\text{Spin}}_{4k-1}(P^{4k-1})$ of degree +2. For each n, Poincaré duality and pulling back from P^{4k-1} to P^n produce the elements $s^n \in \Omega^0_{\text{Spin}}(P^n)$ with augmentation +2. To emphasize the fact that $s^n \neq 2 \cdot 1$ we point out the following.

(2.3) THEOREM. Under the kO-orientation morphism

$$\Omega^*_{\mathrm{Spin}}(P^n) \to kO^*(P^n)$$

 $s^{n} \rightarrow 1 + \eta$, where η is the canonical line bundle.

The following, on the other hand, is straightforward geometry.

(2.4) Proposition. For any integer j there is a map

$$h': (V_q^{4k-1,n}, \partial V_q^{4k-1,n}) \to (M_q^{4k-1,n}, \partial M_q^{4k-1,n})$$

of degree q+2j such that the Poincaré dual of $[V_q^{4k-1,n}, \partial V_q^{4k-1,n}; h']$ is $b_q^n+j\cdot s^n\in\Omega^0_{\mathrm{Spin}}(P^n)$.

3. The Brieskorn normal invariants. In [4] we pointed out how the Brieskorn examples Q_{2d+1}^n produce smooth normal invariants of P^n . Here we show how these examples lead to elements of $[P^n, SG/Spin]$ compatible with the machinery of the previous section.

As remarked in [4], each Q_{2d+1}^{4k+1} is homotopy equivalent to P^{4k+1} . Consequently the map

$$h_{2d+1}: (V_{2d+1}^{4k+3,4k+1}, \partial V_{2d+1}^{4k+3,4k+1}) \to (M_{2d+1}^{4k+3,4k+1}, \partial M_{2d+1}^{4k+3,4k+1})$$

of degree +1 provided by (2.4) is a homotopy equivalence of Spin manifolds. Now the construction of [6] produces in our case a "classifying" SG/Spin-bundle over $M^{4k+3,4k+1} \simeq P^{4k+1}$ for h_{2d+1} , i.e. an element $a_{2d+1}^{4k+1} \in [P^{4k+1}, SG/Spin]$. Restriction to $P^n \subset P^{4k+1}$ defines the element $a_{2d+1}^n \in [P^n, SG/Spin]$.

Now we follow the description of $[P^n, \mu]$ as given in §1. We find that a_{2d+1}^n leads to the element $b_{2d+1}^n - d \cdot s^n \in \Omega^0_{\mathrm{Spin}}(P^n)$ which by (2.2) and (2.3) is sent (under kO-orientation) to the element $1 + d \cdot \xi \in kO^0(P^n)$, where $\xi = 1 - \eta$. This gives our main result on the Brieskorn examples.

(3.1) THEOREM. Under kO-orientation we have

$$[P^{n}, \mu](a_{2d+1}^{n}) = 1 + d \cdot \xi$$

as a virtual line bundle over Pⁿ.

Numerous results follow from (3.1), in particular the following which stems from (1.3).

(3.2) THEOREM. The natural maps

$$[P^n, \mu]: [P^n, SG/\mathrm{Spin}] \to [P^n, BO],$$

 $[P^n, \nu]: [P^n, G/O] \to [P^n, BSO],$

are epimorphisms of groups (where BO and BSO have their tensor product structures).

Results of Browder [3] can be applied to show that the epimorphisms in (3.2) are canonically split—in fact, that $[P^n, \nu]$ classifies the Brieskorn examples of smooth normal invariants for P^n . Similarly, $[P^n, \mu]$ classifies the Brieskorn elements $a_{2d+1}^n \in [P^n, SG/Spin]$. Browder's results alone do not give this except for $n \le 5$.

4. Numerical results. By results of Adams [1], $kO^0(P^n) = KO(P^n)$

is the commutative ring generated by 1 and $\xi = 1 - \eta$ subject to the relations $\xi^2 = 2 \cdot \xi$ and $a_{n+1} \cdot \xi = 0$, where a_k is given by the table:

Now it is clear that the multiplicative group of virtual line bundles $1+d\cdot\xi\in[P^n,\ BO]$ is isomorphic to $Z_2\times Z_{a_{n+1}/2}$. Moreover, the two generators $1-\xi=\eta$ and $1-2\cdot\xi=2\cdot\eta-1$ of $[P^n,\ BO]$ generate the Z_2 and $Z_{a_{n+1}/2}$ factors corresponding to $[P^n,\ P^\infty]$ and $[P^n,\ BSO]$, respectively—as indicated in the following diagram:

$$[P^n, P^\infty] \rightleftarrows [P^n, BO] \rightleftarrows [P^n, BSO]$$
 $\parallel \rangle \qquad \parallel \rangle \qquad \parallel \rangle$
 $Z_2 \rightleftarrows Z_2 \times Z_{a_{n+1}/2} \rightleftarrows Z_{a_{n+1}/2}$

- (4.1) THEOREM. There are $a_{n+1}/2$ distinct Brieskorn smooth normal invariants of P^n .
- (4.2) COROLLARY. There are $a_{4k+2}/2 = 2^{2k}$ smoothly distinct Brieskorn homotopy projective (4k+1)-spaces. For k>0, these yield only 4 combinatorially distinct homotopy projective (4k+1)-spaces.
- (4.3) COROLLARY. For $n \not\equiv 1 \mod 4$ and n > 5, there are $a_{n+1}/4$ smoothly distinct homotopy projective n-spaces which yield only 2 combinatorially distinct homotopy projective n-spaces.

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