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MINIMAL TRANSFORMATION GROUPS WITH DISTAL POINTS

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The purpose of the present note is to announce results concerning the structure of transformation groups (hereinafter called flows) which are minimal and have distal points. Detailed proofs will appear in [9].

Fixed for the discussion is a group T, and we write $\mathfrak{X}$, $\mathfrak{Y}$, $\cdots$, $\mathfrak{Z}$ for flows (T, X), (T, Y), $\cdots$, (T, Z). All phase spaces X, Y, $\cdots$, Z are to be compact and metrizable. Below, $d(\cdot, \cdot)$ will denote a compatible metric for X. $\mathfrak{X}\xrightarrow{\pi}\mathfrak{Y}$ will be used to indicate a *homomorphism* of flows. That is, $\pi: X\rightarrow Y$ is continuous, surjective, and π is equivariant ($\pi T = T\pi$). $\mathfrak{Y}$ is said to be a *factor* of $\mathfrak{X}$.

Given a flow $\mathfrak{X}$, a point $x\in X$ is *distal* for $\mathfrak{X}$ provided $\inf_{t\in T} d(tx, ty) \neq 0$, $y\neq x$. $\mathfrak{X}$ is *point-distal* if there exists a distal point $x\in X$ with dense orbit. Point-distal flows are minimal [1], [6] (the phase space has no proper, closed, invariant subset), and every factor of a point-distal flow is point-distal.

Our first theorem settles a question raised by Knapp [6]. A flow is *nontrivial* if its phase space has more than one point.

THEOREM 1. *Every nontrivial point-distal flow has a nontrivial equicontinuous factor.*

In preparation of Theorem 2 we recall the notion of an isometric extension which is due to Furstenberg [4]. Let $\mathfrak{X}\xrightarrow{\pi}\mathfrak{Y}$ be a homomorphism of flows, and let $S\subseteq X\times X$ be the set

$$S = \{(x_1, x_2) \mid \pi x_1 = \pi x_2\}.$$

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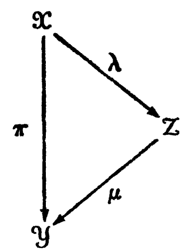
$\mathfrak{X}$ is an *isometric extension* of $\mathcal{Y}$ if there exists a continuous function R on S which defines a metric on each fiber $X_y = \pi^{-1}y$, $y \in Y$, and such that $R(tz) = R(z)$, $t \in T$, $z = (x_1, x_2) \in S$. (The latter condition is meaningful because by the equivariance of π, $TS = S$.) $\mathfrak{X}$ is a *proper* isometric extension of $\mathcal{Y}$ if X_y does not reduce to a point for some y. (For an isometric extension of a minimal flow either all X_y reduce to a point or else none do.)

THEOREM 2. *Let* $\mathfrak{X} \xrightarrow{\pi} \mathcal{Y}$ *be a homomorphism of point-distal flows. In addition, suppose*

(i) *For some* $y \in Y$, X_y *has a dense subset consisting of distal points.*

(ii) π *is an open map.*

If the fiber X_y *in* (i) *does not reduce to a point, there exists a proper isometric extension,* $\mathcal{Z}$, *of* $\mathcal{Y}$ *and homomorphisms* λ *and* μ *such that the triangle*



is commutative.

Theorem 2 is a generalization of Theorem 10.3 of [4]. There, $\mathfrak{X}$ is minimal distal, (i) is trivially satisfied, and (ii) is a consequence of distality. In the present context (i) will be true, for example, if the set of distal points for $\mathfrak{X}$ is a residual subset of X. Condition (ii) is generally not true, however, and the essential reason for this is that there is a second building block for point-distal flows.

DEFINITION 1. Let $\mathfrak{X} \xrightarrow{\pi} \mathcal{Y}$ be a homomorphism of minimal flows. $\mathfrak{X}$ is said to be an *almost automorphic extension* of $\mathcal{Y}$ if for some $y \in Y$ the fiber X_y $(= \pi^{-1}y)$ reduces to a point.

The terminology in Definition 1 is motivated by the study of "almost automorphic" flows made in [8]. To paraphrase in terms of Definition 1, the assertion of §3.4 of [8] is that every minimal almost automorphic flow is an almost automorphic extension of an equicontinuous flow.

The map π of Definition 1 is "almost one-to-one" in the following sense. There is a residual set $Y' \subseteq Y$ (indeed a dense G_δ) such that $X_{y'}$ is a singleton for each $y' \in Y'$.

The following theorem has nothing to do with point-distality except as a tool, and is perhaps of independent interest.

THEOREM 3. *Let $\mathcal{X} \xrightarrow{\pi} \mathcal{Y}$ be a homomorphism of minimal flows. There exist canonically determined minimal flows $\mathcal{X}^*$ and $\mathcal{Y}^*$ and a commutative diagram*

$$(1)\qquad \begin{array}{ccc} \mathcal{X} & \xleftarrow{\ p\ } & \mathcal{X}^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^*} \\ \mathcal{Y} & \xleftarrow{\ q\ } & \mathcal{Y}^* \end{array}$$

such that

(a) *$\mathcal{X}^*$ and $\mathcal{Y}^*$ are almost automorphic extensions of $\mathcal{X}$ and $\mathcal{Y}$, respectively.*

(b) *π^* is an open map.*

Theorem 3 applies in particular to homomorphisms of point-distal flows. Suppose $\mathcal{X} \xrightarrow{\pi} \mathcal{Y}$ is such a homomorphism, and suppose there is a distinguished fiber X_y as in assumption (i) of Theorem 2. It turns out that in (1) the sets $q^{-1}y = \{y^*\}$ and $p^{-1}x = \{x^*\}$, $x \in X_y$, are all singletons. Moreover, if $x \in X_y$ is distal, so is x^*. Finally, we have the equality

$$X^*_{y^*} = \pi^{*-1} y^* = p^{-1} X_y$$

and $p: X^*_{y^*} \to X_y$ is a homeomorphism. It follows that $X^*_{y^*}$ has a dense subset consisting of distal points, and therefore the homomorphism $\mathcal{X}^* \xrightarrow{\pi^*} \mathcal{Y}^*$ satisfies both conditions (i) and (ii) of Theorem 2.

With notations as above, suppose X_y does not reduce to a point. (If it does, then $\mathcal{X}^* = \mathcal{X} = \mathcal{Y}^*$.) Then $X^*_{y^*}$ does not reduce to a point, and Theorem 2 applies. We obtain a proper isometric extension, $\mathcal{Z}^*$, of $\mathcal{Y}^*$, homomorphisms λ and μ, and setting $r = q \circ \mu$, a commutative diagram:

$$(2)\qquad \begin{array}{ccc} \mathcal{X} & \xleftarrow{\ p\ } & \mathcal{X}^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \lambda} \\ \mathcal{Y} & \xleftarrow{\ r\ } & \mathcal{Z}^* \end{array}$$

$\mathcal{Z}^*$ is a proper isometric extension of an almost automorphic extension of $\mathcal{Y}$.

DEFINITION 2. A minimal flow $\mathfrak{X}$ is an *AI extension* of a flow $\mathcal{Y}$ if $\mathfrak{X}$ is an isometric extension of an almost automorphic extension of $\mathcal{Y}$. $\mathfrak{X}$ is a *proper AI* extension if the isometric extension is proper.

One of the earliest symbolic minimal sets was the Morse minimal set [7] (T=integers). The Morse flow is point-distal, and the representation of it given in §3 of [10] is, in our present terminology, as an AI extension of the equicontinuous flow $x \to x+1$ on the 2-adic completion of the integers.

DEFINITION 3. A flow $\mathfrak{X}$ is an *AI flow* if there exists an ordinal α and an inverse system $\{\mathfrak{X}_\beta\}_{\beta \leq \alpha}$, $\{p_{\beta\delta}\}_{\delta \leq \beta \leq \alpha}$ satisfying the conditions

(A) $\mathfrak{X}_0$ is trivial and $\mathfrak{X}_\alpha = \mathfrak{X}$.

(B) $\mathfrak{X}_{\beta+1}$ is a proper AI extension of $\mathfrak{X}_\beta$, $\beta+1<\alpha$, an AI extension if $\beta+1=\alpha$.

(C) If $\beta \leqq \alpha$ is a limit ordinal, then $\mathfrak{X}_\beta = \lim^{-1}_{\delta<\beta} \mathfrak{X}_\delta$.

Because our phase spaces are compact metric, the ordinal α in Definition 3 must be less than the first uncountable ordinal. To obtain Furstenberg's definition of a quasi-isometric flow it is only necessary to replace "AI" by "isometric" in (B).

Using Theorems 2 and 3 and transfinite recursion, one can then prove

THEOREM 4. *Let $\mathfrak{X}$ be a point-distal flow whose set of distal points is a residual subset of X. Then $\mathfrak{X}$ has an almost automorphic extension which is an AI flow.*

It is not hard to see that the set of distal points for an AI flow is a dense G_δ and hence residual. It follows that any flow $\mathfrak{X}$ for which the conclusion of Theorem 4 is true will have its set of distal points residual. Because of this, and because the known examples of point-distal flows obey the structure theorem, we make the

CONJECTURE. *Let $\mathfrak{X}$ be a point-distal flow (with compact metric phase space). The set of distal points for $\mathfrak{X}$ is a residual subset of X.*

If our conjecture is true, it is possible that the easiest proof will be by means of an alternative proof of the structure theorem. The existence of a direct proof is, however, considered more likely.

For the proofs of Theorems 1, 2, and 4 we employ techniques of Furstenberg [4] and Ellis [2], [3], roughly as follows. Let $\mathfrak{X}$ be a point-distal flow, and let $E=E(\mathfrak{X})$ be the Ellis semigroup of $\mathfrak{X}$[2]. (E, a semigroup of transformations of X, is the closure in X^X of the range of T under the mapping $t \to \{t_x\}_{x \in X}$, $t_x = tx$.) If $\mathfrak{X}$ is *distal*, then E is a group (Ellis). In this situation Furstenberg defines a (weaker) compact, T_1 topology on X, with respect to which the transformations of E are continuous (i.e., homeomorphisms). With the point-open topology E is a compact, T_1 space, and multiplication is separately continuous, inversion continuous. In our situation E is not usually a group, and so we choose any minimal left ideal $I \subseteq E$ [2], and use the Ellis decomposition

$$I = \bigcup_{u \in J} G_u \tag{3}$$

where $J = J(I)$ is the set of idempotents in I and $G_u = uI$, $u \in J$. G_u is a group with identity u for each u, and the union (3) is disjoint [2]. Next, define X_u for one fixed $u \in J$ by $X_u = \{x \in X \mid ux = x\}$. Unless the point-distal flow is actually distal, X_u is a proper dense subset of X. G_u is a group of (generally discontinuous) one-to-one transformations of X_u. We define a "Furstenberg topology" on X_u, making X_u a compact, T_1 space. Then with the point-open topology G_u is a compact, T_1 space with multiplication separately continuous, inversion continuous. In the Furstenberg topology G_u is a group of homeomorphisms of X_u. While certain modifications are necessary, the analysis from this point is very similar to the analysis in [4].

It is interesting to note the statement which corresponds to Furstenberg's remark that his topology is T_2 if and only if the minimal distal flow is equicontinuous. In the context of point-distal flows the Furstenberg topology will be T_2 if and only if the flow is almost automorphic; i.e., if and only if it is an almost automorphic extension of an equicontinuous flow. The Furstenberg topology can be defined for an arbitrary minimal flow, but there is in general no implication to be drawn from its being T_2.

Ellis uses the decomposition (3) in determining the structure of "group-like extensions" of minimal sets [3]. There, I is a "universal" minimal set, and Ellis defines a topology on a fixed G_u making it compact, T_1 with multiplication separately continuous, inversion continuous. The relationship between the Ellis and Furstenberg topologies is not clear, but the former does yield a proof of the Furstenberg structure theorem. Our Theorems 1, 2, and 4 do not follow from the work of Ellis because a homomorphism of point-distal flows, even if open, rarely (except when the flows are distal) defines a group-like extension.

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