ON THE LAWS OF CERTAIN LINEAR GROUPS

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In recent years a great deal of attention has been devoted to the study of finite nonabelian simple groups, but one aspect which seems to have been little considered is that of the laws which they satisfy. In a recent paper, two of the present authors gave a basis for the laws of PSL(2, 5) (Cossey and Macdonald [1]), and here we present bases for the laws of PSL(2, 7) and PSL(2, 9). An important feature of the proofs of these results is the knowledge of a two variable basis for the laws of S_4 (the symmetric group of degree 4), which we also state. We show how bases for certain SL(2, p) can be derived from bases for the corresponding PSL(2, p). Finally, we give some laws which hold in PSL(2, p^n) (p a prime) in more general cases.

In notation and terminology we follow the book of Hanna Neumann [2]. We would draw the reader's attention particularly to the law v_n and its properties, given in 52.31 and 52.32 of [2]. Also note that var G denotes the variety generated by a group G.

Our results are as follows.

THEOREM 1. *A basis for the laws of* var S_4 *is*

(1) $x^{12}=1$,

(2) $[x^6, y^6]=1$,

(3) $[x, y]^6=1$,

(4) $[x^2, y^2]^2=1$,

(5) $[x^3, y^3, y^3]^3=1$,

(6) $[[x, y]^3, y^3, y^2]=1$.

THEOREM 2. *A basis for the laws of* var PSL(2, 7) *is*

(1) $x^{84}=1$,

(2) $\{(x^{21}y^{21})^{52}(x^{21}y^{63})^4[x^{21}, y^{42}]^9\}^{21}=1$,

(3) $\{(x^{28}y^{28})^{57}[x^{28}, y^{28}]^{14}\}^{28}=1$,

(4) $[x^3, y^{12}]^{12}=1$,

(5) $\{[[y^{-24}x^{21}y^{24}, y^{-12}x^{21}y^{12}]^{37}, x^{21}]^{13}y^{-48}[x^{21}, y^{48}]^7y^{48}\}^{21}=1$,

(6) $\{[y^{-12}x^{28}y^{12}, x^{28}]^6[x^{28}, y^{36}]^2\}^{28}=1$,

(7) $\{(x^7y^7)^{12}(x^7y^{49})^{12}[x^{35}, y^{49}]^{11}\}^{42}=1$,

(8) $\{\{(x^{77}y^{77})^{72}[x^7, y^7]^{15}[x^7, y^{77}]^3\}^{36}[x^{42}, y^{42}]^5\}^7=1$,

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(9) $\{(x^{21}y^{63})^{72}(x^{28}y^{21})^{24}[[[x^7, y^7]^{21}, y^{21}]^7, y^{14}]\}^7=1$,
(10) $v_{168}=1$.

THEOREM 3. *A basis for the laws of* var PSL(2, 9) *is*
(1) $x^{60}=1$,
(2) $\{(x^{15}y^{15})^{16}(x^{15}y^{45})^{16}[x^{15}, y^{30}]^{37}\}^{15}=1$,
(3) $\{(x^{20}y^{20})^{21}[x^{20}, y^{20}]^{29}\}^{20}=1$,
(4) $\{\{(x^{12}y^{12})^5(x^{12}y^{48})^5\}^{19}[x^{12}, y^{12}]^{16}\}^{36}=1$,
(5) $\{[x^{36}y^{40}x^{24}, y^{20}]^3[y^{40}, x^{36}]^2\}^{20}=1$,
(6) $\{[y^{40}x^{36}y^{20}, x^{12}]^5[y^{40}, x^{36}]^4\}^{12}=1$,
(7) $\{x^{48}[y^{45}x^{12}y^{15}, x^{24}]^5x^{12}[x^{12}, y^{30}]^{18}\}^{12}=1$,
(8) $\{\{[x^{30}, y^{36}]^{45}(x^{15}y^{12})^{21}(x^{15}y^{48})^{57}\}^{10}[x^{15}, y^{12}]^{45}\}^{15}=1$,
(9) $\{(x^5y^5)^{12}[x^5, y^5]^{30}\}^5=1$,
(10) $\{(x^5y^5)^{12}[x^{30}, y^{30}]^5\}^{15}=1$,
(11) $\{(x^5y^5)^{12}[[[x^5, y^5]^{15}, y^5]^{25}, y^{10}]^{25}\}^5=1$,
(12) $v_{360}=1$.

For the remainder of this paper, p will always denote a prime.

THEOREM 4. *Let* $w(x_1, \cdots, x_n)=1$ *be a basis for the laws of* var PSL(2, p), *where* $p=8h\pm 1$, *with* h *odd, or* $p=8h\pm 3$. *Then a basis for the laws of* var Sl(2, p) *is*
(1) $[w(x_1, \cdots, x_n), y]=1$,
(2) $(w(x_1, \cdots, x_n))^2=1$.

THEOREM 5. (a) *The following law holds in* PSL(2, p^n), $p\neq 2, 5$;

$$[x^s, y^s]^s = 1 \quad \textit{if } p^n \equiv 3 \bmod 4,$$

and

$$\{(x^sy^s)^\alpha[x^s, y^s]^\beta\}^s = 1 \quad \textit{if } p^n \equiv 1 \bmod 4,$$

where $s=1/4(p^{2n}-1)$,

$$\alpha \equiv 0 \bmod p, \quad \alpha \equiv 1 \bmod s,$$

and

$$\beta \equiv -1 \bmod p, \quad \beta \equiv 0 \bmod s, \quad \textit{or} \quad \beta \equiv -1 \bmod ps.$$

(b) *The following law holds in* PSL(2, p^n), $p\geqq 7$;

$$[x^s, y^t]^s = 1 \quad \textit{if} \quad p^n \equiv 3 \bmod 4,$$

$$\{[(x^s)^{y^t}, x^s]^\alpha[x^s, y^t]^\beta\}^s = 1,$$

if $p^n\equiv 1 \bmod 4$, *where* $s=1/4(p^{2n}-1)$,

$$r \mid p+1,\ r \nmid (p-1)/2, \quad t = (p(p^{2n}-1))/4r,$$
$$\alpha \equiv 0 \bmod p, \quad \alpha \equiv -1 \bmod s, \quad \beta \equiv 1 \bmod p, \quad \beta \equiv 0 \bmod s.$$

These laws imply that finite groups satisfying such laws have abelian-Sylow p-subgroups, and that an element of order r which belongs to the normaliser of a p-group belongs to its centraliser.

A full account of these and related results and their proofs will be published soon.

REFERENCES

1. John Cossey and Sheila Oates Macdonald, *A basis for the laws of* PSL(2, 5), Bull. Amer. Math. Soc. 74 (1968), 602–606.

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