ON THE BOUNDARY POINT PRINCIPLE FOR ELLIPTIC EQUATIONS IN THE PLANE[1]

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1. Let D be an open connected subset of E^n $(n \geqq 2)$ and denote by $\mathfrak{L}_\alpha(D)$ the class of second order uniformly elliptic operators of the form $L = \sum_{i,j=1}^n a_{ij}\partial^2/\partial x_i \partial x_j$ with coefficients defined in D and satisfying there the condition $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geqq \alpha \sum_{i=1}^n \xi_i^2$, for some constant α in the range $0 < \alpha \leqq 1/n$, and the normalization $\sum_{i=1}^n a_{ii} = 1$. It is well known [1]–[4] that such differential operators enjoy the following strong minimum and boundary point principles: A nonconstant, twice differentiable function $u(x)$, satisfying $Lu \leqq 0$ in D, cannot attain a local minimum in D. Moreover if u attains a local minimum at a boundary point x^0 where ∂D has the inner sphere property, and if ν is a unit vector directed internally to the sphere, then

$$\liminf_{t \to 0^+} \left\{ \frac{u(x^0 + t\nu) - u(x^0)}{t} \right\} > 0.$$

Equivalently, the boundary point principle states that for $\|x - x^0\|$ sufficiently small there exists a *positive* constant m (depending upon ν) such that

$$u(x) \geqq u(x^0) + m\|x - x^0\|$$

along the line $x^0 + t\nu$. In this note we wish to obtain, for the case of a plane domain $(n = 2)$, an analogous lower bound for the approach of $u(x)$ to a minimum occurring on the boundary when the inner sphere property is replaced by an inner cone (sector) property. The proof is based upon a comparison with a barrier function which has recently been obtained [5] for the class $\mathfrak{L}_\alpha$ in a plane sector with the aid of elliptic extremal operators [6]. Our result is the best possible for the class of differential operators $\mathfrak{L}_\alpha$ and moreover shows explicitly the dependence upon the ellipticity constant α.

2. We shall first describe our barrier function for the plane sector

$$S(\theta_0) = \{(x, y) : r > 0, \ |\theta| < \theta_0 < \pi\}$$

where r, θ denote the polar coordinates of the point (x, y).

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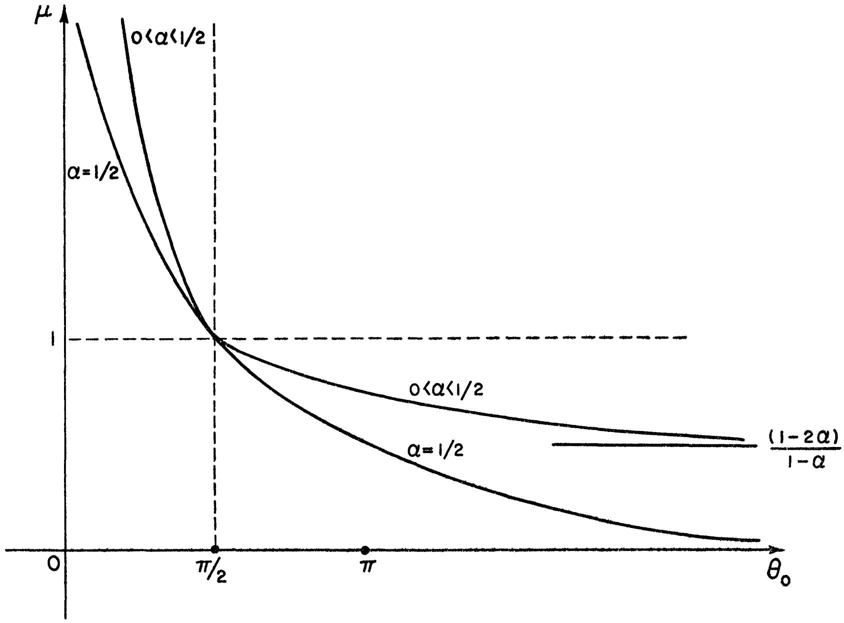

FIGURE 1

For the class of differential operators $\mathfrak{L}_\alpha$ with $0<\alpha<\frac{1}{2}$ we define the constants $\zeta_1=\cos^{-1}(1-2\alpha)\in(0,\pi/2)$, $\zeta_2=\pi-\zeta_1$, and

$$\mu = \frac{2(1-2\alpha)}{\cos\zeta+(1-2\alpha)} \tag{2.1}$$

where:

(i) $\zeta\in(\zeta_1,\zeta_2)$ is the solution of $\zeta\tan\zeta_1/\tan\zeta+\zeta_2=2\theta_0$ if $0<2\theta_0\leqq\pi$; or

(ii) $\zeta\in(0,\zeta_1)$ is the solution of $(\pi-\zeta)\tan\zeta_1/\tan\zeta+\zeta_1=2\theta_0$ if $\pi<2\theta_0<2\pi$.

For the class $\mathfrak{L}_{1/2}$, which consists of the single operator $\{\partial^2/\partial x^2+\partial^2/\partial y^2\}/2$, we define $\mu=\pi/2\theta_0$.

A sketch of μ as a function of α and θ_0 is shown in Figure 1. Note that μ is a monotone decreasing function of θ_0 and $\mu(\alpha,\pi/2)=1$.

Next we define the periodic function $C(\theta;\theta_0;\alpha)$ in the parametric form

$$\begin{aligned} C(\theta;\theta_0;\alpha) &= \frac{\cos\varphi\{1-\nu_1\cos\varphi\}^{(|\mu-1|-1)/2}}{\{1-\nu_2\cos\varphi\}^{(|\mu-1|+1)/2}} \\ \theta &= \frac{(4\alpha(1-\alpha))^{1/2}}{\mu}\int_0^\varphi \frac{d\xi}{(1-\nu_1\cos\xi)(1-\nu_2\cos\xi)} \end{aligned} \tag{2.2}$$

where

$$\nu_1 = \frac{(|\mu - 1| - 1)(1 - 2\alpha)}{\mu} \quad \text{and} \quad \nu_2 = \frac{(|\mu - 1| + 1)(1 - 2\alpha)}{\mu}.$$

The following properties of $C(\theta; \theta_0; \alpha)$ are easily established:

$$(2.3)\quad \begin{aligned} &\text{(a) } C(\theta; \theta_0; 1/2) = \cos \mu\theta = \cos(\pi\theta/2\theta_0);\\ &\text{(b) } C(\theta; \pi/2; \alpha) = (\cos\theta)/(4\alpha(1-\alpha))^{1/2};\\ &\text{(c) } C(\theta; \theta_0; \alpha) > 0 \quad \text{for } |\theta| < \theta_0 \quad \text{and} \quad C(\pm\theta_0; \theta_0; \alpha) = 0;\\ &\text{(d) } C(\theta; \theta_0; \alpha) \sim \cos(\pi\theta/2\theta_0) \quad \text{as } \theta \to \pm\theta_0. \end{aligned}$$

The function

$$(2.4)\qquad v(x, y) = r^{\mu} C(\theta; \theta_0; \alpha),$$

positive in the sector $S(\theta_0)$ and vanishing on its sides, is the barrier which we seek. It has been obtained in [5] as a solution of the minimizing equation relative to the class $\mathfrak{L}_\alpha$. It follows from the theory of extremal operators [6] that for every operator $L \in \mathfrak{L}_\alpha$ we have

$$Lv(x, y) \geqq 0 \qquad \forall\, (x, y) \in S(\theta_0).$$

Furthermore there exists an operator $L' \in \mathfrak{L}_\alpha$ such that $L'v = 0$ in $S(\theta_0)$.

3. We now state our main result.

THEOREM. *Let D be an open subset of the plane and let $u(x, y)$ be a nonconstant, twice differentiable function in D which is continuous on $\bar{D}$ and satisfies $Lu \leqq 0$ in D for some $L \in \mathfrak{L}_\alpha$. Suppose that u attains a local minimum of u_0 at a boundary point P_0 which subtends an open truncated sector S, $S \subset D$, of half angle θ_0. Then there exists a neighborhood Ω of P_0 and a positive constant m such that*

$$(3.1)\qquad u(x, y) \geqq u_0 + mr^{\mu} C(\theta; \theta_0; \alpha) \text{ in } \bar{S} \cap \bar{\Omega},$$

where r, θ are polar coordinates measured from the vertex and axis of S and μ, C are defined by (2.1) *and* (2.2) *respectively.*

PROOF. Since the class $\mathfrak{L}_\alpha$ is invariant under translation or rotation of coordinates there is no loss of generality in assuming P_0 to be the origin and the axis of S to be the x-axis so that r, θ become the usual polar coordinates.

By the hypotheses there exists an $R > 0$ such that $u \geqq u_0$ in $[S(\theta_0)]^- \cap \{r \leqq R\}$. Since $u(x, y)$ is not identically constant we conclude from the strong minimum principle, the boundary point principle, and the property (2.3)(d) that

$$m = R^{-\mu} \inf_{S(\theta_0)\cap\{r=R\}} \left\{\frac{u(x, y) - u_0}{C(\theta; \theta_0; \alpha)}\right\} \tag{3.2}$$

is a positive constant. Let us define

$$w(x, y) = u(x, y) - u_0 - mv(x, y)$$

in $[S(\theta_0)]^- \cap \{r \leqq R\}$, where v is our barrier given by (2.4).

In $S(\theta_0) \cap \{r < R\}$ we have $Lw = Lu - mLv \leqq 0$ while $w = (u - u_0) \geqq 0$ on $\partial S(\theta_0) \cap \{r \leqq R\}$ and, by (3.2), $w \geqq 0$ on $S(\theta_0) \cap \{r = R\}$. It follows from the minimum principle that $w \geqq 0$ in $[S(\theta_0)]^- \cap \{r \leqq R\}$, which is the desired result.

REMARKS. (1) Since the barrier function $v = r^\mu C(\theta; \theta_0; \alpha)$ is itself a solution of $L'u = 0$ in $S(\theta_0)$ for some $L' \in \mathfrak{L}_\alpha$ our result cannot be improved.

(2) Note that $\mu = 1$ for $\theta_0 = \pi/2$, $\mu > 1$ for $\theta_0 < \pi/2$, and $\mu < 1$ for $\theta_0 > \pi/2$. Thus for $\theta_0 = \pi/2$ our result coincides with that of the boundary point principle. When $\theta_0 < \pi/2$ the difference quotient $(u - u_0)/r$ may tend to zero when $r \to 0$, as is well known for domains without the inner sphere property. Note however that for $\theta_0 > \pi/2$ this difference quotient is unbounded as $r \to 0$ so that the theorem implies that no interior directional derivative can exist at a local minimum occurring at the vertex of an obtuse angle.

4. Suppose now that the hypotheses of the theorem hold for an operator L^* of the form

$$L^* = L + b_1\partial/\partial x + b_2\partial/\partial y$$

with $L \in \mathfrak{L}_\alpha$ and $(b_1^2 + b_2^2)^{1/2} = o\{1/r\}$ as $r \to 0$, where r denotes distance measured from the boundary point P_0.

Let a fixed $\epsilon > 0$ be given and denote by Ω^* the corresponding neighborhood of P_0 in which $(b_1^2 + b_2^2)^{1/2} \leqq \epsilon/r$. Using again the minimizing operator relative to the class $\mathfrak{L}_\alpha$ it may be shown that there exists a function $T_\epsilon(\theta)$, with properties similar to those of (2.3)(c) and (d), such that $v_\epsilon = r^{\mu+\epsilon} T_\epsilon(\theta)$ is a barrier for L^* in $S \cap \Omega^*$. It follows that our theorem may be extended to the operator L^*, with the conclusion (3.1) replaced by

$$u(x, y) \geqq u_0 + mr^{\mu+\epsilon} T_\epsilon(\theta) \quad \text{in } \bar{S} \cap \bar{\Omega}. \tag{4.1}$$

The results (3.1) or (4.1) are also valid for the operators $L + c$ or $L^* + c$, respectively, if $c \leqq 0$ in a neighborhood of P_0, provided that we assume that the minimum value u_0 is negative. ($u_0 \leqq 0$ is sufficient if in addition the growth of $|c|$ is suitably limited near P_0; e.g., if c is bounded below.)

5. Our results may also be applied to certain quasilinear equations. As an example let us consider a nonconstant solution $\phi(x, y)$ of the equation of minimal surfaces

$$(1 + \phi_y^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (1 + \phi_x^2)\phi_{yy} = 0$$

in the sector $S(\theta_0)$. If the gradient of ϕ is bounded

$$|\operatorname{grad} \phi| \leqq M \tag{5.1}$$

then ϕ satisfies the linear equation $L\phi = 0$, where

$$L = \{(1 + \phi_y^2)\partial^2/\partial x^2 - 2\phi_x\phi_y\partial^2/\partial x\partial y + (1 + \phi_x^2)\partial^2/\partial y^2\} / \{2 + |\operatorname{grad} \phi|^2\}$$

is in the class $\mathfrak{L}_\alpha$ with

$$\alpha = 1/(M^2 + 2). \tag{5.2}$$

If ϕ achieves a local minimum of ϕ_0 at the origin then in a neighborhood of the origin we have, from (3.1) and (5.1),

$$\phi_0 + mr^\mu C(\theta; \theta_0; \alpha) \leqq \phi(x, y) \leqq \phi_0 + Mr, \qquad m > 0, \tag{5.3}$$

where μ and C are defined by (2.1) and (2.2). Note that if $\theta_0 > \pi/2$ then (5.3) yields a contradiction since $\mu < 1$. As a result we may state: A nonconstant minimal surface with bounded gradient cannot attain a local minimum at the vertex of an obtuse angle.

References

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