THE IMPOSSIBILITY OF FILLING E^n WITH ARCS

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The purpose of this paper is to outline a proof of the following

MAIN THEOREM. *If f is a closed continuous map of E^n onto any space S, then some point in S has an inverse image which is not an arc.*

In 1936 J. H. Roberts [1] showed that there does not exist an upper semicontinuous (usc) collection of arcs filling the plane. Recently L. B. Treybig [2] has obtained some partial results for polygonal arcs in E^n. In 1955 Eldon Dyer [3] outlined a proof that there is no continuous decomposition of E^n into arcs. This proof incorporates some of the ideas of both Roberts and Dyer.

We will suppose that all statements are for E^n for a given n.

DEFINITIONS. If U and V are sets with disjoint closures, we say that an arc α has *k folds* between U and V if α contains $k+1$ disjoint subarcs between U and V. Furthermore, if the distance between each pair of the $k+1$ subarcs is greater than ϵ, we say that the *width* of the folds is greater than ϵ. If α contains a subarc which has endpoints in U and which intersects V, then α is said to have a fold with the *bend* in V.

If K is a set, $\epsilon > 0$, let $N_\epsilon(K)$ denote the open ϵ-neighborhood of K in E^n. If H is a collection of sets, let H^* denote the set of all points covered by elements of H.

Suppose A is compact and B is a closed subset of A. If any two points of $E^n - A$ which are separated by A are also separated by B, then B is said to be *essential* in A. If H is a usc collection of arcs and points filling A and B intersects each element of H, then B is said to be *full* in A^H. If B meets each element of H in a continuum, then B is said to be a *quasi-section* of A^H.

Assume H is a usc collection of arcs and points filling the compact set X.

LEMMA 1. *If Y is a quasi-section of X^H then Y is essential in X.*

The proof is an exercise in the Vietoris mapping theorem on the Čech homologies of X, Y, and the decomposition space.

LEMMA 2. *If K is full in X^H, U is open, $\overline{U} \cap K = \varnothing$, and no element*

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of H has a fold between K and U with bend in U, then X has an essential subset that misses U.

It follows from the hypothesis that for each $h \in H$, $h-(h\cap U)$ contains a unique component which intersects K. If Y is the union of all such components, Y is a quasi-section of X^H and hence Y is an essential subset that misses U.

REMARK. Obviously, under the above conditions if U is connected, U cannot intersect two distinct components of $E^n - X$.

Suppose G is a usc collection of arcs and points filling some complete metric space. The collection G is said to be *continuous* at an element g if for every finite chain $\mathcal{H}$ of open sets covering g, there exists an $\epsilon>0$ such that each element of G contained in $N_\epsilon(g)$ intersects each element of $\mathcal{H}$. The collection G is said to be *equicontinuous* at g if G is continuous at g and no element of G contained in $N_\epsilon(g)$ contains a fold between two nonadjacent links of $\mathcal{H}$. Roberts proved that the set G_1 of elements at which G is continuous is dense in G, and the set G_2 of elements at which G_1 is equicontinuous is dense in G_1.

Suppose X is compact and H is a usc collection of arcs and points filling X.

LEMMA 3. *If K is full in X^H, Q is a quasi-section of X^H that contains an endpoint of each element of H, and $K\cap Q=\emptyset$, then there is a quasi-section Y of X^H such that Q is not contained in Y.*

Let h_2 denote an element of H_2. We can find an $\epsilon>0$ such that h_2 contains no folds between $N_{2\epsilon}(K)$ and $N_\epsilon(Q)$. Since H_1 is equicontinuous at h_2, if $h_1 \in H_1$ is near h_2, then h_1 contains no folds between $N_{2\epsilon}(K)$ and $N_\epsilon(Q)$. Suppose $h\in H$, h is very near h_2, and the component of $h-[h\cap N_\epsilon(K)]$ that meets Q contains a fold between $N_{2\epsilon}(K)$ and $N_\epsilon(Q)$. This implies that every element of H_1 very near h must also contain such a fold, since every such element must span between K and Q, and to do this it must "follow" h from $N_\epsilon(Q)$ to $N_{2\epsilon}(K)$, back to $N_\epsilon(Q)$, and again to $N_{2\epsilon}(K)$ before it can intersect K. Hence from Lemma 2 we have a quasi-section Y_1 of X^H of arcs from Bd $N_\epsilon(K)$ to Q, and a quasi-section Y_2 of Y_1 which misses a very small open set about $h_2\cap Q$. Trivially, Y_2 is a quasi-section of X^H, and this completes the proof of Lemma 3.

We will suppose throughout the remainder of the paper that G is a usc collection of arcs filling E^n.

Suppose g is an element at which G is continuous, U and V are open sets with disjoint closures, and each of U and V contains an endpoint of g. Let K denote a closed neighborhood of the endpoint of g in U, $K \subset U$. Let M denote the set of all elements of G which intersect Bd $N_\epsilon(g)$, for some small ϵ. Hence M^* is compact and if ϵ was selected small enough, then (a) $M^* \cap K$ is full in $M^{G|M}$, and (b) V meets two components of $E^n - M^*$. The remark following Lemma 2 implies there is an arc with a fold between $K \cap M$ and V, and hence

THEOREM 1. *There exists an element of G with a fold between U and V with the bend in V.*

To prove the Main Theorem, we need to find some arc with infinitely many folds between U and V. It should be noted that it is insufficient to obtain a sequence $\{\alpha_i\}_{i>0}$ of elements of G such that each α_j contains j folds between U and V, since the limit of such a sequence may be an arc with no folds at all. Thus we need sequences $\{\alpha_i\}_{i>0}$ of arcs of G and $\{d_i\}_{i>0}$ of positive numbers such that for each j, if $k>j$, α_k contains j folds between U and V of width at least d_j. The limit of such a sequence would be an arc with infinitely many folds between $\bar{U}$ and $\bar{V}$. The following is an analogue to a lemma of Roberts.

THEOREM 2. *There exists an open set W such that each element of G that meets W contains a fold between $\bar{U}$ and $\bar{V}$.*

REMARK. For $\epsilon>0$ the set of all elements having a fold between $\bar{U}$ and $\bar{V}$ of width $\geqq \epsilon$ is closed. Thus using Theorem 2 and the Baire category theorem we easily obtain an open set W' and a positive number d such that each element of G that meets W' contains a fold of width greater than d. The proof then proceeds similar to that of Roberts.

The proof of Theorem 2 is crucial and requires more machinery.

Note that since U and V were selected arbitrarily it is sufficient to show that for $\epsilon>0$, there is an open set W such that each element that meets W contains a fold between $N_\epsilon(U)$ and $N_\epsilon(V)$.

From Theorem 1 there is some arc α of G which contains a fold between U and V with the bend in V. Let α' denote a subarc of α which has a fold between U and V with the bend in V but no subarc of α' has this property. Hence α' minus its endpoints separates α into two components K_1 and K_2. Let U_1 and U_2 denote small disjoint open sets about K_1 and K_2 respectively. Thus $\alpha \cap U \subset U_1 \cup U_2$, and every element of G near α meets either U_1 or U_2 since every such element of G must intersect U (every element near α is also near the element g at which G is continuous). Hence if δ is small enough, every element of G contained in $N_\delta(\alpha)$ which meets both U_1 and U_2 contains a fold between $N_\epsilon(U)$ and V. Thus the following lemma implies Theorem 2.

Suppose $\alpha \in G$, U_1 and U_2 are open sets, each containing an endpoint of α, $\epsilon > 0$, and every element of G near α meets either U_1 or U_2.

LEMMA 4. *There exists an open set W such that every element of G that meets W intersects both $N_\epsilon(U_1)$ and $N_\epsilon(U_2)$.*

We assume this lemma to be false.

Let G' denote the usc collection of arcs and points filling E^n such that $g' \in G'$ if and only if either (a) for some element g of G, g' is a component of $g - \{g \cap [N_{\epsilon/2}(U_1) \cup N_{\epsilon/2}(U_2)]\}$ or (b) g' is a point of $N_{\epsilon/2}(U_1) \cup N_{\epsilon/2}(U_2)$. Trivially, if Lemma 4 is false for G then it is false for G'.

Let L denote the set of all elements of G' which are near α (here α is considered as a point set, since $\alpha \notin G'$) and which fail to intersect both $N_\epsilon(U_1)$ and $N_\epsilon(U_2)$. L is naturally divided into L_1, those elements intersecting $N_\epsilon(U_1)$ but not $N_\epsilon(U_2)$, and L_2, those elements intersecting $N_\epsilon(U_2)$ but not $N_\epsilon(U_1)$.

Let M denote $B^n - L^*$, where B^n is a very large ball containing g, α, L^*, U, V, etc.

LEMMA 4.1. *M separates U_1 from U_2.*

This is obvious since $L^* = L_1^* \cup L_2^*$.

LEMMA 4.2. *L^* is dense near α.*

This is a direct result from the assumption that Lemma 4 is false.

LEMMA 4.3. *No element of G' near α contains a fold between $N_\epsilon(U_1)$ and $N_\epsilon(U_2)$.*

Suppose $\beta' \in G'$ near α such that β' contains a fold between $N_\epsilon(U_1)$ and $N_\epsilon(U_2)$ with the bend in $N_\epsilon(U_2)$. Let $\beta \in G$ such that $\beta' \subset \beta$. If W is a very small open set about a point in the bend of β in $N_\epsilon(U_2)$, then every element of G that meets W must "follow" β into $N_\epsilon(U_1)$, since recall that every element near α must intersect either U_1 or U_2. We are assuming such a W does not exist and hence this completes Lemma 4.3.

Let Q denote $\overline{L_1^*} \cap \overline{L_2^*}$.

LEMMA 4.4. *Every essential subset of M contains Q.*

Every subset of M that fails to contain every point of Q fails to separate L_1^* from L_2^* and hence fails to separate U_1 from U_2.

LEMMA 4.5. *Q is a quasi-section of $M^{G'|M}$ near α.*

If $g' \in G'$, and p is a point of $g' \cap Q$, then there is a subarc l_p of g' between Bd $N_{\epsilon/2}(U_1)$ and Bd $N_{\epsilon/2}(U_2)$ containing p. We observe that l_p is the union of a segment in $\overline{L_1^*}$ and a segment in $\overline{L_2^*}$. Thus if q is another point of $g' \cap Q$, since g' contains no fold between $N_\epsilon(U_1)$ and $N_\epsilon(U_2)$, then $l_p = l_q$, and hence p and q lie in the same component of $g' \cap Q$. Thus Q is a quasi-section of $M^{G'|M}$ near α.

By a similar argument, if we let X denote $\overline{L_1^*} \cap M$ near α, then $M' =$ [Closure of ($M -$ elements of G' near α)] $\cup X$ is a quasi-section of $M^{G'|M}$. Hence M' is an essential subset of M and contains Q by Lemma 4.4. It follows from the definition of X that X is compact, $G'|X$ is a usc collection of arcs filling X such that Bd $N_{\epsilon/2}(U_1)$ is full in $X^{G'|X}$, and Q is a quasi-section of X which contains an endpoint of each arc in the decomposition. Lemma 3 implies there is a quasi-section Y of X which does not contain Q. However, if $M'' =$ cl $(M' - X) \cup Y$, M'' is a quasi-section of M' and hence a quasi-section of $M^{G'|M}$. Thus M'' is an essential subset of M which does not contain Q. This contradicts Lemma 4.4 and completes the proof of Theorem 2.

References

1. J. H. Roberts, *Collections filling a plane*, Duke Math. J. **2** (1936), 10–19.

2. L. B. Treybig, *Concerning upper semicontinuous decompositions of E^n whose nondegenerate elements are polyhedral arcs or starlike continua*, (to appear).

3. *Summary of Lectures and Seminars*, Summer Institute on Set Theoretic Topology, Univ. of Wisconsin, Madison, Wisconsin, 1955, p. 105.

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