ON MANIFOLDS WITH INVOLUTION

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We consider a smooth involution ω on a smooth closed n-manifold M from the bordism point of view, as in [2, Chapter IV]. We know that the fixed-point set of ω is the disjoint union of submanifolds; let k be the maximum dimension of these. It is clear that if ω is free, M bounds a 1-disk bundle over the orbit space. Now fix k, and let n, M, and ω vary. Conner and Floyd prove [2, Theorem (27.1)] that if M does not bound, n cannot be arbitrarily large. Their proof is nonconstructive, and fails to give an upper bound for n. We obtain the precise bound.

THEOREM 1. *Suppose k is the maximum dimension of the fixed-point submanifolds of the smooth involution ω on the closed nonbounding n-manifold M. Then $n \leqq 5k/2$ (if k is even) or $n \leqq (5k-1)/2$ (if k is odd). Further, if we are given that the unoriented cobordism class of M is indecomposable, then $n \leqq 2k+1$.*

Examples in the extremal dimensions are easily constructed. Take homogeneous coordinates $(x_0, x_1, x_2, \cdots, x_i, x_1', x_2', \cdots, x_i')$ on real projective $2i$-space P_{2i} $(i>0)$, and define the involution ω_i by

$$\omega_i(x_0, x_1, \cdots, x_i, x_1', \cdots, x_i') = (x_0, x_1, \cdots, x_i, -x_1', \cdots, -x_i').$$

Then the product involution $\omega_i \times \omega_j$ on $P_{2i} \times P_{2j}$ maps the hypersurface $H_{2i,2j}$ defined by the equation

$$x_0 y_0 + x_1 y_1 + \cdots + x_i y_i + x_1' y_1' + \cdots + x_i' y_i' = 0$$

into itself, where for clarity we take coordinates $(y_0, y_1, \cdots, y_j, y_1', \cdots, y_j')$ on P_{2j}, and assume $i \leqq j$. The fixed-point dimension of $\omega_i \times \omega_j | H_{2i,2j}$ is found to be $i+j-1$. The manifold $H_{2i,2j}$ has dimension $2i+2j-1$ and its cobordism class $[H_{2i,2j}]_2$ is indecomposable if and only if the binomial coefficient

$$\binom{i+j}{i}$$

is odd. We can always choose i and j satisfying this condition and $i+j=m$ whenever m is not a power of 2. As an example for the first assertion of Theorem 1, we take the product of many copies of the 5-dimensional example $(H_{2,4}, \omega_1 \times \omega_2 | H_{2,4})$, with possibly one copy of (P_2, ω_1).

We deduce Theorem 1 from Theorems 2 and 3 below, which are purely algebraic. They concern the bordism J-homomorphism

$$J_m: \mathfrak{N}_i(BO(m)) \to \mathfrak{N}_{i+m-1}(BO(1))$$

defined in [2, §25]. (It has only a tenuous connection with the Hopf-Whitehead J-homomorphism, which could be written

$$J: \pi_i(BO(m)) \to \pi_{i+m-1}(P_m).)$$

Let us recall from [2] the bordism classification of manifolds with involution (M, ω). The normal bundle in M of the i-dimensional fixed-point set of ω determines an element $\nu_i \in \mathfrak{N}_i(BO(n-i))$. The main structure theorem (28.1) asserts that these elements characterize the bordism class of (M, ω), and are arbitrary, subject only to the condition

$$\sum_i J_{n-i}\nu_i = 0.$$

We stabilize J_n by defining a homomorphism

$$J: \mathfrak{N}_*(BO) \to \mathfrak{N}[[\theta]],$$

where $\mathfrak{N}[[\theta]]$ is the ring of homogeneous formal power series over $\mathfrak{N}$ in an indeterminate θ of degree -1. Given $\alpha \in \mathfrak{N}_i(BO)$, we put $J\alpha = \sum \alpha_r \theta^r$. To define the coefficient $\alpha_r \in \mathfrak{N}_r$, we first lift α to $\alpha' \in \mathfrak{N}_i(BO(m))$ for some $m \geqq r-i+1$, and put $\alpha_r = \epsilon\Delta^{i+m-r-1}J_m\alpha'$, where Δ is the bordism Smith homomorphism defined in [2, §26], and $\epsilon: \mathfrak{N}_*(BO(1)) \to \mathfrak{N}$ is the canonical augmentation; α_r is independent of m by Theorem (26.4) of [2]. It is more natural to define $J: F \to \mathfrak{N}[[\theta]]$, where $F = \oplus_i \mathfrak{N}_i(BO)$, by extending linearly. Then the elements ν_i, when included in $\mathfrak{N}_*(BO)$, may be added in F to form an element $\nu \in F$. The relation $\sum_i J_{n-i}\nu_i = 0$, and also (24.2) of [2], are combined in the following formula.

THEOREM 2. *$J\nu = [M]_2\theta^n +$ terms with higher powers of θ.*

Now the cross product of vector bundles makes F into an ungraded polynomial ring. It is easy to see that

$$J1 = 1 + [P_2]_2\theta^2 + [P_4]_2\theta^4 + [P_6]_2\theta^6 + \cdots.$$

Therefore we define $J': F \to \mathfrak{N}[[\theta]]$ by setting $J'\alpha = (J\alpha)\cdot(J1)^{-1}$, so that $J'1 = 1$. We may clearly replace J by J' in Theorem 2. We are interested in the case when

$$\nu \in F_k = \oplus_{i=0}^{i=k} \mathfrak{N}_i(BO) \subset F.$$

THEOREM 3. *J': $F \to \mathfrak{N}[[\theta]]$ is a ring homomorphism. Further, we can find systems of polynomial generators $z_i \in \mathfrak{N}_i$ for $\mathfrak{N}$ and x_i for F, for each i not of the form $2^q - 1$, such that*

(a) *$J'x_i = z_i\theta^i +$ terms with higher powers of θ,*

(b) *If we assign to x_i the weight $i/2$ (i even) or $(i-1)/2$ (i odd), then F_k consists of all polynomials of weight $\leqq k$ in the elements x_i.*

There ought to be a direct geometric proof that J' is a ring homomorphism.

The computation of J_n, and hence of J', is in principle known from [2, Chapter IV]. All that is lacking is a certain amount of technique. Full details will appear in [1].

REFERENCES

1. J. M. Boardman, *Unoriented bordism and cobordism*, (to appear).

2. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Springer-Verlag, Berlin, 1964.

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