THE RADIAL HEAT EQUATION WITH POLE TYPE DATA[1]

BY L. R. BRAGG

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1. **Introduction.** Recently, detailed studies have been undertaken relating to the solutions and expansions of solutions of the initial value problem

$$(1)\quad \begin{aligned} &\text{(a)}\quad U_t(r, t) = \Delta_\mu U(r, t), \qquad r > 0,\ t > 0,\\ &\text{(b)}\quad U(r, 0) = \phi(r) \end{aligned}$$

with $\Delta_\mu \equiv D_r^2 + [(\mu-1)/r]D_r$. Results have been obtained when $\phi(r)$ is entire of growth $(1, \sigma)$ in r^2 [1], [3], [4] and these have been extended to the L_2 theory in [3]. In this note, we state some results on the structures of solutions of (1) when the data function $\phi(r)$ has a pole at $r=0$ but is otherwise entire. These structures are defined in terms of convolution integrals and the proofs are based on the Laplace transform formulation [2] of solutions of (1) and the expansion theory referred to above. The details of the proofs will appear in a forthcoming paper that will also discuss logarithmic singularities.

We denote by $U^\mu(r, t; \phi(r))$ the solution of (1) defined by

$$\int_0^\infty K_\mu(r, \xi; t)\phi(\xi)d\xi$$

with

$$K_\mu(r, \xi; t) = \frac{1}{2t} r^{1-\mu/2}\xi^{\mu/2} \exp\left[-(r^2+\xi^2)/4t\right] I_{\mu/2-1}(r\xi/2t).$$

(See [1], [4].) The abbreviation $a = r^2/16t^2$ will be used in the statement of results.

2. **Main results.** Our first theorem relates to functions $\phi(r)$ that are odd while the remaining results relate strictly to functions with poles.

THEOREM 1. *Let $\phi(r) = r\psi(r)$ in which $\psi(r)$ is an entire function of r^2 of growth $(1, \sigma)$. For $0 \leqq t < 1/4\sigma$ and $\mu > 2$,*

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$$U^\mu(r,\,t;\,\phi(r))$$
$$= \frac{r^{2-\mu}\exp[-r^2/4t]}{\pi^{1/2}(4t)^{5/2-\mu}}\int_0^a (a-\xi)^{-1/2}\xi^{(\mu-3)/2}e^{4\xi t}U^{\mu-1}(4t\xi^{1/2},\,t;\,\psi)d\xi. \tag{2}$$

THEOREM 2. *Let $\phi(r)=r^{2-\mu-2\alpha}\psi(r)$ with $0\leqq\alpha<1/2$ and $\psi(r)$ an entire function of r^2 of growth $(1,\sigma)$. For $0\leqq t<1/4\sigma$ and $\mu>2$*

$$\begin{aligned} U^\mu(r,\,t;\,\phi(r)) &= \frac{r^{2-\mu}\exp[-r^2/4t](4t)^{\mu/2-\alpha-1}}{\Gamma(\mu/2+\alpha-1)} \\ &\quad\cdot\int_0^a \xi^{-\alpha}(a-\xi)^{\mu/2+\alpha-2}e^{4\xi t}U^{2-2\alpha}(4t\xi^{1/2},\,t;\,\psi)d\xi. \end{aligned} \tag{3}$$

Observe that the choice $\alpha=0$ in Theorem 2 corresponds to the case in which the multiplier of $\psi(r)$ is precisely the potential function for the Laplacian operator Δ_μ. This theorem shows that the pole can be more badly behaved than the potential function. In fact, the following theorem shows that the pole can be as badly behaved as $r^{-\mu+\epsilon}$ for arbitrary $\epsilon>0$ and still give rise to a classical solution.

THEOREM 3. *Let $\phi(r)=r^{2-\mu-2\alpha}\{A+r^2\psi(r)\}$ in which α is close to but less than 1, $\mu/2+\alpha>2$, A is a constant, and $\psi(r)$ is an entire function of r^2 of growth $(1,\sigma)$. For $0\leqq t<1/4\sigma$,*

$$\begin{aligned} U^\mu(r,\,t;\,\phi) &= \frac{r^{2-\mu}\exp[-r^2/4t]}{(4t)^{1-\mu/2}} \\ &\quad\cdot\left\{\frac{A(4t)^{1-\alpha}}{\Gamma(\mu/2+\alpha-1)}\int_0^a \xi^{-\alpha}(a-\xi)^{\mu/2+\alpha-2}e^{4\xi t}d\xi+\frac{(4t)^{2-\alpha}}{\Gamma(\mu/2+\alpha-2)}\right. \\ &\quad\left.\cdot\int_0^a \xi^{1-\alpha}(a-\xi)^{\mu/2+\alpha-3}e^{4\xi t}U^{4-2\alpha}(4t\sqrt{\xi},\;t;\;\psi)d\xi\right\}. \end{aligned} \tag{4}$$

It follows, from the change of valuables $\xi=a\sigma$, that

$$\lim_{r\to 0;\,t>0} U^\mu(r,t;\,\phi)$$

exists in all of the above theorems. This simply states that the pole in the data function dissipates from the solution function.

Finally, as a corollary to Theorem 2 where $\alpha=0$, we have the special result:

COROLLARY 2.1. *Let $\mu=2m$ be an even integer with $m\geqq 2$. Then*

$$
\begin{aligned}
U^{2m}(r,\ t;\ r^{2-2m+2j}) &= r^{2-2m}R_j^{4-2m}(r,\ t)+(-1)^{j+1}\exp\left[-r^2/4t\right] \\
&\cdot\sum_{k=0}^{m-2-j}\frac{(m-2-k)!}{k!(m-2-j-k)!}\, r^{2(1-m+k)}(4t)^{j-k},\ 0\leqq j\leqq m-2.
\end{aligned}
\tag{5}
$$

In this, $R_\mu^{4-\mu}(r,\ t)=j!(4t)^jL_j^{(1-\mu/2)}(-r^2/4t)$ with $L_j^\nu(x)$ the generalized Laguerre polynomial of degree j and index ν. In the case that μ is even with $\mu\geqq 4$, we can divide the data into the pole type terms (finite in number) and the entire part. The corollary applies to the pole terms while the expansion theory in [1], [4] applies to the entire part.

References

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CASE INSTITUTE OF TECHNOLOGY AND
OAKLAND UNIVERSITY