SUBGROUPS OF FINITE GROUPS[1]

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1. Introduction. Let G be a finite group. What can we say about G if we are given some information about the subgroups of G? That is, what does the local structure of G tell us about the global structure of G? In this paper we will describe some answers to this question and some remaining unsolved cases. We will concentrate on the following special case:

Problem 1. Given a particular subgroup H of G, what can we say about G?

Throughout this paper, G will denote an arbitrary finite group, and all groups considered will be finite.

2. Centralizers of involutions. Suppose τ is an element of G. Let $C(\tau)$ be the centralizer of τ, i.e., the set of elements of G that commute with τ. Many of the answers to Problem 1 in recent years have concerned the case in which $H=C(\tau)$ and τ is an *involution*, that is, an element of order two. Why are elements of order two different from elements of odd prime order? The reason is a paradox: involutions occur in both the hardest groups and the easiest groups with which we have to deal.

In many problems about finite groups, the hardest cases are the nonsolvable groups and, in particular, the simple groups. (We say that G is simple if it is *not* abelian and if it has no normal subgroups other than itself and 1, the identity subgroup. Thus we exclude the cyclic groups of prime order.) The celebrated theorem of Feit and Thompson [7] asserts that if G is not solvable, then the order, or number of elements, of G is even. Thus G has a nonidentity Sylow 2-subgroup, which, in turn, must contain an involution.

THEOREM 1 (FEIT-THOMPSON). *If G is not solvable, then G contains an involution.*

Thus, if G is not solvable, involutions are available. But this does not guarantee that we can handle them; here is where the "easiest" groups come in. Suppose we want to study the local properties of G in some abstract, general way. Nothing could be more local than an

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individual element of G. Take a nonidentity element x of G. By definition, x generates a cyclic subgroup of G. This subgroup is abelian, solvable, and so forth, but it usually does not tell us much about G. So we ask how the elements of G interact with each other. Take another nonidentity element y of G, and let H be the subgroup of G that is generated by x and y. What local information about G do we learn from H?

Perhaps the most important fact we learn is that we may have no local information at all. Although G may be large, complicated, and nonsolvable, H may be equal to G. In fact, every known finite simple group is generated by two elements. (This was shown for almost all known simple groups by Steinberg [18].) But there is one exception to this general chaos, namely, when x and y are involutions. In this case, the equations

$$x^{-1}(xy)x = yx = y^{-1}x^{-1} = (xy)^{-1}$$

and

$$y^{-1}(xy)y = yx = y^{-1}x^{-1} = (xy)^{-1}$$

show that xy generates a subgroup of index two in H. Thus H is a solvable group and is "almost" Abelian. In fact, if xy has order two, H is Abelian and is called a (Klein) *four-group*; otherwise, H is called a *dihedral* group, because it is isomorphic to the group of symmetries of a regular polygon.

Using this property of involutions and an ingenious counting argument, Brauer and Fowler [5] proved one of the strongest answers to Problem 1 yet obtained. Let us denote the order of any subgroup H of G by $|H|$.

THEOREM 2 (BRAUER-FOWLER). *Suppose G is simple and τ is an involution in G. Let $H=C(\tau)$. Then*

$$|G| < |H|^2! = |H|^2(|H|^2 - 1) \cdots 2 \cdot 1.$$

Thus, given H, there are only a *finite* number of possibilities for G (up to isomorphism). Hence H almost *determines* G.

Theorem 2 gives us not only an answer to Problem 1 but also a theoretical program for determining every finite simple group by means of centralizers of its involutions, which, presumably, have a less complicated structure. In fact, many authors have selected various choices of H and have determined all the corresponding groups G. In many cases there is only one group G, and in some cases there are none.

3. Sylow 2-subgroups. Assume G has even order. Let τ be an involution in G and S a Sylow 2-subgroup of G that contains τ. Suppose τ lies in the center of S, $Z(S)$, i.e., suppose τ commutes with every element of S. Then S is a Svlow 2-subgroup of $C(\tau)$. Thus, given $C(\tau)$, we know the structure of S. But what can we say about G if we know only S?

In general, S tells us much less than $C(\tau)$ tells us. For example, in his thesis [8], Fowler showed that if G is simple and $C(\tau)$ is a four-group, then G is isomorphic to the alternating group on five letters, so $|G| = 60$. In contrast, there are infinitely many simple groups G in which S is a four-group. Some progress has been made, however. Brauer [3] has proved that for infinitely many 2-groups S there exist some, but only finitely many, simple G. A landmark in the classification of simple groups is the determination, by Gorenstein and Walter [14], of every simple group G for which S is a four-group or a dihedral group. Recently, substantial progress has been made in the determination of the simple groups with Abelian Sylow 2-subgroups.

Some negative results have also been obtained. For example, the Feit-Thompson Theorem tells us that a Sylow 2-subgroup of a simple group cannot be the identity group. By a theorem of Burnside ([15, p. 203]), G is not simple if S is a nonidentity cyclic group. Brauer and Suzuki (independently) proved [3] that G is not simple when S is a generalized quaternion group, that is, a noncyclic 2-group with only one involution.

The latter two results illustrate some similar but "dual" methods of proving that a group G is not simple. Both theorems are proved by showing that G has a factor of prime order in a composition series for G. Burnside's theorem establishes sufficient conditions for G to have a normal subgroup N such that G/N has prime order. We might describe this approach as "working from the top down," i.e., from G to N. In contrast, the approach of Brauer and Suzuki might be described as "working from the bottom up." Since they are concerned with the simplicity of G, they assume that G has no normal subgroup of odd order except the identity subgroup. Then they show that G has a normal subgroup of order two and thus is not simple.

It is easy to see that a normal subgroup of order two in a group G must be contained in $Z(G)$, the center of G. In certain cases, we can use the following result to locate elements of S that lie in $Z(G)$.

THEOREM 3 [11]. *Let S be a Sylow 2-subgroup of G and let $x \in S$. Suppose G has no normal subgroup of odd order except the identity sub-*

group. Then $x \in Z(G)$ *if and only if* x *satisfies the following uniqueness condition*:

(U) *Whenever* $g \in G$ *and* $g^{-1}xg \in S$, *then* $g^{-1}xg = x$.

In other words, $x \in Z(G)$ if and only if x is the only conjugate of itself that lies in S. For example, if S is a generalized quaternion group and x is the unique involution in S, then x satisfies (U) since every conjugate of an involution is an involution. Thus we obtain the Brauer-Suzuki Theorem; this is not surprising, however, since the Brauer-Suzuki Theorem was needed to prove Theorem 3. More generally, using Theorem 3 we may show that G cannot be simple if S has the form $Q \times R$, where Q is a generalized quaternion group, R is a finite 2-group, and every involution of R lies in the center of R. In particular, G is not simple if S is a direct product of any number of generalized quaternion groups.

In spite of these applications, Theorem 3 arose as a problem about loops and groups of odd order [9]. Suppose $|G|$ is odd. For every $x \in G$ there exists a unique element $x^{1/2}$ of G such that $(x^{1/2})^2 = x$. We define a new operation, $\circ$, on G by $x \circ y = x^{1/2}yx^{1/2}$. Thus $x \circ y = xy$ if G is Abelian. In fact, if G is nilpotent of class two, G forms an Abelian group under $\circ$. In any case, G forms a loop under $\circ$, that is, for arbitrary $a, b \in G$ there exist unique solutions of the equations $x \circ a = b$ and $a \circ y = b$, and G has an identity element (namely, 1). Moreover, for $x \in G$, $x \circ x^{-1} = x^{-1} \circ x = 1$. Thus elements of G have inverses under $\circ$.

A simple calculation shows that for $x, y, z \in G$,

$$x \circ (y \circ (x \circ z)) = (x \circ (y \circ x)) \circ z \tag{L1}$$

and

$$(x \circ y)^{-1} = x^{-1} \circ y^{-1}. \tag{L2}$$

Now, subgroups of G form subloops of G under $\circ$. It is possible that G contains other subloops, but the fact that $x \neq x^{-1}$ for $x \neq 1$ insures that every subloop of G has odd order. Theorem 3 is equivalent to the proposition that every loop of odd order that satisfies (L1) and (L2) occurs as a subloop of G for some group G of odd order. From the Feit-Thompson Theorem, this is equivalent to the proposition that every such loop is solvable, that is, its composition factors are groups of prime order. (A normal subloop is defined to be the kernel of a loop homomorphism, and other terms are defined as in group theory. The Jordan-Hölder Theorem holds for loops; see [6, p. 67].)

Although it would be interesting to prove Theorem 3 or the Brauer-

Suzuki Theorem by loop theory, the proof of Theorem 3 was entirely group-theoretical. It requires some arguments similar to those of Brauer and Fowler, which rest upon the fact that two distinct involutions generate a four-group or a dihedral group. Since these theorems seem plausible for arbitrary primes, some questions arise.

Problem 2. Does Theorem 2 hold for some function of $|H|$ with τ of arbitrary prime order? Similarly, does Theorem 3 hold when p is an odd prime, S is a Sylow p-subgroup of G, and 1 is the only normal subgroup of G whose order is not divisible by p?

Problem 3. Can a simple group have a Sylow 2-subgroup of the form $Q\times R$ for some generalized quaternion group Q?

A first step toward solving Problem 2 has been taken by E. Shult [**16**]:

THEOREM 4 (SHULT). *Let p be a prime and S a Sylow p-subgroup of G. Let $x\in S$. Assume that*

(a) *x satisfies (U), and*

(b) *whenever x normalizes a subgroup H of G of order relatively prime to p, then x centralizes H.*

Then $x\in Z(G)$.

4. **Automorphism groups.** In this section we do not assume that G is simple. Let N be a normal subgroup of G. For each element g of G we define an automorphism $\phi(g)$ of N by the rule $x\rightarrow g^{-1}xg$. It is not difficult to show that ϕ is a homomorphism of G into Aut N, the automorphism group of N, and that ϕ maps N onto In N, the group of inner automorphisms of N. The kernel of ϕ is the centralizer, $C(N)$, of N, i.e., the set of elements of G that commute with every element of N. Thus, information about Aut N will give us information about G. Since Aut N is partly determined by the automorphism groups of the composition factors of N, we are led back to the case of a simple group. The outstanding conjecture about automorphism groups is the following:

SCHREIER'S CONJECTURE. *If G is simple, then* Aut $G/$In G *is a solvable group.*

Although this conjecture has been verified for every *known* simple group (a unified proof for most cases was given by Steinberg [**17**]), the general case is far from being solved. However, here, too, the structure of a particular subgroup of G can give us a great deal of information. Brauer [**4**] has proved that G must satisfy Schreier's Conjecture if its Sylow 2-subgroups lie in a certain class of finite 2-groups. By applying Theorem 3, we may extend Brauer's results. Let us denote by $N(S)$ the normalizer of S, i.e., the set of elements g in G such that $gS=Sg$.

THEOREM 5 [12]. *Let S be a Sylow 2-subgroup of G. If* $\operatorname{Aut} S$ *is solvable and G is simple, then* $\operatorname{Aut} G/\operatorname{In} G$ *is solvable. G satisfies Schreier's Conjecture if S satisfies any of the following conditions*:

(a) *S is generated by two elements.*

(b) *S is generated by three elements and $N(S)/C(S)$ is not a 2-group.*

(c) *The involutions in S commute with each other and are all conjugate in G.*

When considering a normal subgroup N of G, one can sometimes assume that $|N|$ and $|G/N|$ are relatively prime integers. A theorem of Schur [15, p. 224] asserts that in this case G must contain a subgroup A such that $G=AN$ and $A\cap N=1$. Here $|A|=|G/N|$, so by the Feit-Thompson Theorem, A is solvable if N is not solvable. We can obtain a great deal of local information in this case. For example, given any prime p, there exists a Sylow p-subgroup S_p of N such that every element of A maps S_p onto itself. Obviously, for each p, A induces a group A_p of automorphisms of S_p; but A_p will not tell us too much about the structure of A if A_p is the identity group. In an attempt to find some sort of bound on the structure of A by using local information, we are led to the following question, raised by Thompson:

Problem 4. Let A be a subgroup of $\operatorname{Aut} G$ such that $|A|$ and $|G|$ are relatively prime. Does there exist a solvable subgroup H of G such that:

(a) every element of A maps H onto itself, and

(b) the only element of A that maps every element of H onto itself is the identity element?

An elementary argument shows that Problem 4 may be answered affirmatively if, whenever G is *simple*, there exists a proper (not necessarily solvable) subgroup H of G that satisfies (a) and (b). But how would we choose H? For each prime p, there must exist Sylow subgroups S_p that satisfy (a), but examples show that even $N(S_p)$ may violate (b). On the other hand, take an involution τ of G. In view of the intimate connection between $C(\tau)$ and G demonstrated by the Brauer-Fowler Theorem, one would suspect that $C(\tau)$ satisfies (b). This is the case; but unfortunately $C(\tau)$ need not satisfy (a).

THEOREM 6 [10]. *Suppose G is simple. Let τ be an involution in G and let α be an automorphism of G such that the order of α is relatively prime to the order of G. If $\alpha\neq 1$, then α moves some element of $C(\tau)$.*

Some restriction on the order of α is necessary in Theorem 6. The inner automorphism $g \to \tau^{-1}g\tau$, $g \in G$, moves an element of G precisely when it is *not* in $C(\tau)$; more generally, for every integer $n > 1$, there exist G, τ, and α such that α has order n, n divides $|G|$, and α moves an element g of G precisely when g is not in $C(\tau)$.

The proof of Theorem 6 requires Theorem 3 and a counting argument involving involutions.

Problem 5. Does Theorem 6 hold for τ of arbitrary prime order?

5. **Normalizers of J-subgroups.** Let x be an element of a Sylow p-subgroup S of G. Consider the uniqueness condition of Theorem 3 (stated there for $p=2$):

(U) Whenever $g \in G$ and $g^{-1}xg \in S$, then $g^{-1}xg = x$.

We have used Theorem 3 to show how a single subgroup of G affects the global structure of G. But (U) itself is a rather "global" condition, so we ask, in turn, if there exists a local characterization of (U). We obtain a rather frustrating answer.

For every finite p-group S, define

$$d(S) = \max \{ |A| : A \text{ an Abelian subgroup of } S \}$$

and let $J(S)$ be the (characteristic) subgroup of S generated by all the Abelian subgroups of order $d(S)$ in S. (This subgroup was introduced by Thompson in [23].) Let S^4 be the symmetric group on four letters; $|S^4| = 24$. We say that a group Q is *involved* in G if Q is isomorphic to H/K for some subgroups H and K of G such that K is a normal subgroup of H.

THEOREM 7. *Let p be a prime. Let x be an element of a Sylow p-subgroup S of G. If x is in the center of $N(J(S))$, then x satisfies* (U), *except possibly when*:

(a) $p=2$;

(b) *for all* $y \in Z(S)$, $y^2 \neq x$; *and*

(c) S^4 *is involved in* G.

Thus, although Theorem 3 has been proved only for $p=2$, we obtain a local characterization of (U) that works only for p odd. Unfortunately, some exceptions for $p=2$ are necessary in Theorem 7; without them, S^4 itself would be a counterexample.

Condition (U) occurred in Theorem 3, which we described as "working from the bottom up." There is an analogue to Theorem 7 that corresponds to the "dual" concept of "working from the top down." Let p be a prime. Define $O^p(G)$ to be the subgroup of G generated by those elements of G whose order is not divisible by p. Then $G/O^p(G)$ is a p-group, and $O^p(G)$ is contained in every normal subgroup N of G for which G/N is a p-group. Hence G has a normal subgroup of index p if and only if $O^p(G)\neq G$.

THEOREM 8. *Let p be a prime such that $p\geqq 7$. Let S be a Sylow p-subgroup of G and let $M=N(J(S))$. Then $G/O^p(G)$ is isomorphic to $M/O^p(M)$.*

COROLLARY. *Let p be a prime such that $p\geqq 7$. Let S be a Sylow p-subgroup of G. If $S\neq 1$ and $N(S)/C(S)$ is a p-group, then $G/O^p(G)\neq 1$.*

For any prime p it may happen that $G/O^p(G)$ is actually isomorphic to S. If this occurs, then $G=SO^p(G)$ and $S\cap O^p(G)=1$, so we say that $O^p(G)$ is a *normal p-complement* for G. The most general result on normal p-complements was obtained by Thompson in a remarkably short paper [21]:

THEOREM 9 (THOMPSON). *Let p be a prime and let S be a Sylow p-subgroup of a finite group G. If $C(Z(S))$ and $N(J(S))$ have normal p-complements, then G has a normal p-complement, except possibly when $p=2$ and S^4 is involved in G.*

Note. Theorem 9 was actually proved in [21] for a slightly different definition of $J(S)$; however, there is only a small difference between the proofs. Although the case $p=2$ was not in the statement of [21], it can be seen from the last step of the proof.

The proof of Theorem 9 depends on several ideas of Thompson that are used in all the theorems in this section (and in Theorem 12 in the next section). For example, we observe that $J(S)$ is a characteristic subgroup of every subgroup of S that contains it; specifically, if $J(S)\subseteq P$, then $J(S)=J(P)$. Similarly, for all Abelian subgroups A and B of S such that $|A|=d(S)$, we obtain $|B(A\cap C(B))|\leqq|A|$. If G is a counterexample to Theorem 7, 8, or 9, we can show that "something goes wrong" in $N(H)$ for some nonidentity p-subgroup H. This follows from a theorem of Burnside [15, p. 46], in Theorem 7; a theorem of Alperin and Gorenstein [2, Theorem A(1)], in Theorem 8; and a theorem of Frobenius [15, p. 217], in Theorem 9. Using a partial ordering introduced by Thompson in [20], we may assume that $G=N(H)$ and that $C(H)\subseteq H$.

At this point, we may assume in Theorem 9 that G is solvable. The remaining part of the proof is subsumed in a general theorem of Thompson about solvable groups [22, Theorem 1]. For p odd, a similar theorem involving only one subgroup (Theorem 12(a), in §6) yields Theorem 10 (below).

In Theorems 7 and 8, we prove that G is "almost" solvable by using a "replacement theorem" of Thompson [23] about the internal structure of p-groups. Then by applying the induction hypothesis and Thompson's partial ordering to the maximal subgroups of G, we complete the proof.

THEOREM 10. *Let p be an odd prime and let S be a Sylow p-subgroup of G. Then G has a normal p-complement if and only if $N(Z(J(S)))$ has a normal p-complement.*

COROLLARY. *Let p be an odd prime and S a p-group. There exists a characteristic subgroup $K(S)$ of S such that:*

(a) *The centralizer of $K(S)$ in S is contained in $K(S)$; and*

(b) *if S is a Sylow p-subgroup of G, then G has a normal p-complement if and only if $N(K(S))/C(K(S))$ is a p-group.*

The subgroups $K(S)$ in the corollary can be obtained from $Z(J(S))$ by defining a certain ascending series of characteristic subgroups of S. For $p \geqq 5$, this is not necessary, however:

THEOREM 11 (THOMPSON). *Let p be a prime such that $p \geqq 5$, and let S be a Sylow p-subgroup of G. Then G has a normal p-complement if and only if $N(J(S))/C(J(S))$ is a p-group.*

The proof of Theorem 11 uses the methods of Theorem 9 and Thompson's recent Replacement Theorem. Theorem 11 is false for $p<5$.

There are numerous possible generalizations of these results. The most obvious gap occurs in the case $p=2$, where Theorem 10 fails rather spectacularly.

Problem 6. Let $p=2$. Assume S^4 is not involved in G. Does Theorem 10 hold for some characteristic subgroup $L(S)$ of S in place of $Z(J(S))$?

If $L(S)$ can be found to satisfy Problem 9 (in §6), then $L(S)$ will satisfy Problem 6. An interesting candidate for $L(S)$ is suggested by Thompson in [23]. One cannot allow arbitrary G in Problem 6; S^4 and the simple group of order 168 must be excluded, as they must be (and are) in every theorem in this section. Other counterexamples show that we must consider subgroups of S that are not even normal in S in order to test for normal 2-complements.

Problem 7. Can anything be said for $p=2$ and G arbitrary?

The following question is somewhat more promising:

Problem 8. Does Theorem 8 hold for some characteristic subgroup $L(S)$ of S in place of $J(S)$ if $p=3$ or $p=5$? Does it hold for $L(S)$ if $p=2$ and if S^4 is not involved in G?

Counterexamples show that we cannot take $L(S)=J(S)$ for $p \leqq 3$ or $L(S)=Z(J(S))$ for $p=2$. A weaker, but still interesting, result would be

$$O^p(G) \cap S = (O^p(C(Z(S))) \cap S)(O^p(N(L(S))) \cap S).$$

If S^4 is not involved in G, this holds for $p=2$ and $L(S)=J(S)$.

6. **Several subgroups.** One of the general problems of group theory is the determination of all finite simple groups. One way to approach this problem is to determine the simple groups that are "small" or "basic" in some sense. Let us say that G is a *minimal simple group* if G satisfies the condition:

(M) G is simple and every proper subgroup of G is solvable.

Just as we learn something about a group from its composition factors, so we may learn something about a group from the minimal simple groups it "contains" as subquotients. These always exist if G is nonsolvable: Let H be a nonsolvable subgroup of least order in G and let K be a maximal normal subgroup of H. Then K is a solvable group and H/K is a minimal simple group.

Thus, if there existed a nonsolvable group of odd order there would exist a minimal simple group of odd order. In the proof of the Feit-Thompson Theorem, the authors take G to be a minimal simple group of odd order and eventually derive a contradiction. As intermediate steps in the proof, they obtain a number of strong properties of the maximal subgroups of G. For example, for certain primes p every Sylow p-subgroup of G is contained in only one maximal subgroup of G.

Gorenstein and Walter [**13**] have isolated several properties of solvable groups that figure crucially in the proof of the Feit-Thompson Theorem. By assuming these properties for certain proper subgroups of a simple group, they have obtained analogues of the intermediate results of Feit and Thompson on maximal subgroups. These analogues apply to finite groups with dihedral Sylow 2-subgroups, which need not satisfy (M).

In another recent breakthrough, Thompson [**24**] has determined the minimal simple groups. More generally, he has determined all the simple N-*groups*. G is called an N-group if it satisfies the condition:

(N) The normalizer of every nonidentity solvable subgroup of G is solvable.

For an odd prime p, one can "localize" some of the properties of the Sylow p-subgroups of an N-group. Let us denote by $\mathrm{SL}(2, p)$ the group of all square matrices of degree two and determinant one whose entries lie in the field of p elements.

THEOREM 12. *Let p be an odd prime, and let S be a Sylow p-subgroup of G. Suppose that for every nonidentity subgroup P of S that contains $Z(S)$, $N(P)$ has no subquotient isomorphic to* $\mathrm{SL}(2, p)$. *Then*:

(a) *If G has a normal p-subgroup P such that $C(P) \subseteq P$, then $Z(J(S))$ is a normal subgroup of G.*

(b) *Two elements of S are conjugate in G if and only if they are conjugate in $N(Z(J(S)))$.*

Now assume that $p \geqq 5$, or that $p = 3$ and G has an Abelian Sylow 2-subgroup. Since SL(2, p) has a non-Abelian Sylow 2-subgroup and is also nonsolvable if $p \geqq 5$, G satisfies the hypothesis of Theorem 12 if G is an N-group. Moreover, if G is actually solvable and has no nonidentity subgroup of order relatively prime to p, then Theorem 12(a) applies.

Let us consider conclusion (b) of Theorem 12 as a proposition about a Sylow p-subgroup S of an arbitrary group G. In the proof of Theorem 12 we show that G satisfies (b) if (b) is satisfied by every subgroup of the form $N(P)$ for $Z(S) \subseteq P \subseteq S$. Recently, Alperin and Gorenstein [2] have generalized this reduction by proving that it is valid for any suitably defined "functor" on p-groups, not just for $Z(J(\ ))$. They have also proved that certain analogous propositions about G hold globally if they hold locally; for example, they show that Theorem 8 may be reduced to the local case. Their results depend on the following theorem of Alperin [1], which, at long last, localizes the problem of conjugacy of elements in a Sylow subgroup.

THEOREM 13 (ALPERIN). *Let p be a prime, and let S be a Sylow p-subgroup of G. Let x and y be elements of S. Suppose $g \in G$ and $g^{-1}xg = y$. Then there exist Sylow p-subgroups $T_1, \cdots, T_n$ of G and elements $g_1, \cdots, g_n, h$ of G such that:*

(a) $g = g_1 \cdots g_n h$;

(b) *$g_i \in N(S \cap T_i)$ for each i, and $h \in N(S)$; and*

(c) *$x \in S \cap T_1$ and $(g_1 \cdots g_i)^{-1}x(g_1 \cdots g_i) \in S \cap T_{i+1}$ for $i = 1, \cdots, n-1$.*

There are many interesting criteria for "basic" simple groups other than (M) and (N). As mentioned earlier, some authors have required that a Sylow 2-subgroup or a centralizer of an involution have a particular form. One of the most general conditions was investigated by Suzuki in a series of papers. He determined [19] all simple groups G that satisfy the condition:

(C) The centralizer of every involution in G has a normal Sylow 2-subgroup.

These results represent some of the recent progress in group theory toward answering Problem 1. We conclude with one elusive and one enormous problem:

Problem 9. Let $p=2$. Assume S^4 is not involved in G. Does Theorem 12(a) hold for some characteristic subgroup $L(S)$ of S in place of $Z(J(S))$?

Problem 10. Find a method of assigning to every simple group G a proper subgroup H that characterizes G in some manner.

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