# DERIVATIONS OF LIE ALGEBRAS[1]

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1. It is known as a theorem of E. Schenkman and N. Jacobson that every nilpotent Lie algebra over a field of arbitrary characteristic has an outer derivation (see [1]). In connection with this theorem, we know the following two types of results, one showing a wider class of Lie algebras which have outer derivations and the other showing the existence of outer derivations in proper ideals of the derivation algebras. Namely, G. Leger [2] has shown that, if a Lie algebra over a field of characteristic 0 whose center is $\neq (0)$ has no outer derivations, it is not solvable and its radical is nilpotent. On the other hand, T. Satô [3] has shown that every nilpotent Lie algebra over a field of characteristic 0 has an outer derivation in the radical of its derivation algebra. We shall generalize and sharpen these results and give more detailed results on outer derivations of Lie algebras over a field of arbitrary characteristic.

2. We denote by $Z(H)$ the center of a Lie algebra $H$. Then we have

THEOREM 1. *Every Lie algebra $L$ over a field $\Phi$ of arbitrary characteristic such that $L \neq L^2$ and $Z(L) \neq (0)$ has an outer derivation. More precisely, such a Lie algebra $L$ has a nilpotent outer derivation $D$ such that $D^2 = 0$, unless $L$ is either 1-dimensional or the direct sum of a 1-dimensional ideal and of an ideal $L_1$ such that $L_1 = L_1^2$ and $Z(L_1) = (0)$.*

In the case where $L$ is not abelian and has no abelian direct summands, take a subspace $M$ of $L$ of codimension 1 containing $L^2$. Then $M$ is an ideal of $L$ and $[L, Z(M)] \neq Z(M)$. Choose an element $e$ of $L$ such that $L = \Phi e + M$ and an element $z$ of $Z(M)$ which is not in $[L, Z(M)]$. Then the endomorphism $D$ of $L$ defined in such a way that $De = z$ and $DM = (0)$ is an outer derivation of $L$ such that $D^2 = 0$.

COROLLARY. *Let $L$ be a Lie algebra over a field of characteristic 0 such that $Z(L) \neq (0)$ and $R$ be the radical of $L$. If $L$ has no outer derivations, $L$ is not solvable and $R = [L, R]$.*

There is another class of nonsolvable Lie algebras which have outer derivations. Namely:

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THEOREM 2. *Let $L$ be a nonsolvable Lie algebra over a field of characteristic* 0 *and $R$ be the radical of $L$. If $R$ has a semisimple outer derivation in the radical of its derivation algebra, then $L$ has a semisimple outer derivation.*

This can be shown by using the following fact: Let $L=S+R$ be a Levi decomposition of $L$. Then among maximal toroidal subalgebras of the radical of the derivation algebra of $R$, there exists one which can be imbedded in the set of all derivations of $L$ which map $S$ into (0). A consequence of the theorem is that if $R$ is nilpotent and has a derivation whose trace is $\neq 0$ then $L$ has a semisimple outer derivation.

3. We shall call a Lie algebra $L$ over a field $\Phi$ to be of type $(T)$ provided

$$L = \Phi e_1 + \Phi e_{1'} + \cdots + \Phi e_n + \Phi e_{n'} + L^2$$

where

$$[e_j, e_k] = [e_{j'}, e_{k'}] = 0, \qquad [e_j, e_{k'}] = \delta_{jk} z \quad \text{with} \quad 0 \neq z \in Z(L),$$

and

$$[e_j, L^2] = [e_{j'}, L^2] = (0) \qquad \text{for } j, k = 1, 2, \cdots, n.$$

We denote by $\mathfrak{D}(L)$ the derivation algebra of a Lie algebra $L$, by $\mathfrak{R}$ the radical of $\mathfrak{D}(L)$, by $\mathfrak{R}_0$ the abelian ideal of $\mathfrak{D}(L)$ consisting of all derivations which map $L$ into $L^2$ and $L^2$ into (0), by $\mathfrak{C}(L)$ the ideal of $\mathfrak{D}(L)$ consisting of all central derivations, and by $\mathfrak{C}_0$ the abelian ideal of $\mathfrak{D}(L)$ consisting of all central derivations which map $Z(L)$ into (0). Then we have

THEOREM 3. *Let $L$ be a Lie algebra over a field of arbitrary characteristic such that $L\neq L^2$ and $Z(L)\neq(0)$.*

(1) *If $L$ is not abelian and has no abelian direct summands and if $L$ is not of type* (T), *then $L$ has an outer derivation in $\mathfrak{R}_0$.*

(2) *Assume that $L$ is not abelian but has an abelian direct summand. If $Z(L)$ is not a direct summand of $L$, then $L$ has an outer derivation in $\mathfrak{R}_0 \cap \mathfrak{C}(L)$. If $Z(L)$ is a direct summand of $L$ and $L/Z(L)$ does not coincide with the derived algebra, then $L$ has an outer derivation in $\mathfrak{C}_0$. If $Z(L)$ is a direct summand of $L$ and $L/Z(L)$ coincides with the derived algebra, then $L$ has a semisimple outer derivation in $\mathfrak{R}$.*

(3) *If $L$ is either abelian or a Lie algebra of type* (T) *such that $L^{(1)}\neq L^{(2)}$, then $L$ has a semisimple outer derivation in $\mathfrak{R}$.*

*In the above statements, $\mathfrak{R}_0$, $\mathfrak{R}_0\cap\mathfrak{C}(L)$, $\mathfrak{C}_0$ and $\mathfrak{R}$ cannot be replaced by any smaller ideals of $\mathfrak{D}(L)$.*

This can be shown by using the following fact: $L$ is of type $(T)$ if and only if $L \neq L^2$, $(0) \neq Z(L) \subset L^2$ and $Z(M) \not\subset L^2$ for every ideal $M$ of $L$ of codimension 1.

COROLLARY 1. *Let $L$ be a Lie algebra over a field of arbitrary characteristic such that $L \neq L^{(1)}$, $L^{(1)} \neq L^{(2)}$ and $Z(L) \neq (0)$. Then $L$ has an outer derivation in $\mathfrak{N}$.*

A solvable Lie algebra $L$ of type $(T)$ is such that $L^2 = \Phi z$ and therefore $L$ is nilpotent and $L^{(1)} \neq L^{(2)}$. Hence by Theorem 3 we have

COROLLARY 2. *Every solvable Lie algebra $L$ over a field of arbitrary characteristic such that $Z(L) \neq (0)$ has an outer derivation in $\mathfrak{N}$. More precisely, if $L$ is not abelian and not of type $(T)$, then $L$ has an outer derivation in $\mathfrak{N}_0$ unless $Z(L)$ is a direct summand of $L$.*

4. Let $L$ be a Lie algebra over a field of characteristic 0. Denote by $A(L)$ the group of all automorphisms of $L$ and by $A_0(L)$ the irreducible component of $A(L)$. We shall call $\sigma \in A(L)$ to be outer provided it does not belong to the smallest algebraic subgroup of $A(L)$ whose Lie algebra contains all inner derivations. Then an application of Theorem 1 is the following:

THEOREM 4. *Let $L$ be a Lie algebra over a field of characteristic 0. If $L \neq L^2$, $Z(L) \neq (0)$ and the Lie algebra of all inner derivations of $L$ is algebraic, then $L$ has an outer automorphism in $A_0(L)$.*

## REFERENCES

1. N. Jacobson, *A note on automorphisms and derivations of Lie algebras*, Proc. Amer. Math. Soc. **6** (1955), 281–283.

2. G. Leger, *Derivations of Lie algebras*. III, Duke Math. J. **30** (1963), 637–645.

3. T. Satô, *On derivations of nilpotent Lie algebras*, Tôhoku Math. J. **17** (1965), 244–249.

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