APPROXIMATION OF BOUNDED FUNCTIONS BY CONTINUOUS FUNCTIONS

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We shall show that every bounded function on a paracompact space has a best approximation by continuous functions, and characterize the functions whose best approximators are unique. This is a special case of a measure-theoretic problem, whose setting is as follows. Let X be a topological space and μ a Borel measure on X which assigns positive mass to each nonempty open set, and has the property that $\mu(Y)=0$ if Y intersects a neighborhood of each point in a μ-null set. The latter condition is automatically fulfilled if each open cover of X has a countable subcover. Let L^∞ be the space of essentially bounded real-valued μ-measurable functions on X, and give it the semi-norm $\|f\|=$essential sup $|f|$. The bounded continuous functions on X form a closed subspace C of L^∞. We say that $g\in C$ is a *best approximator* to $f\in L^\infty$ if $\|f-g\|=\text{dist}(f, C)=\inf\{\|f-h\|: h\in C\}$.

If $f\in L^\infty$ and $x\in X$, $f^*(x)=\lim\ \sup_{y\to x} f(y)=\inf\{$ess sup of f over U: U is a neighborhood of $x\}$; $f_*=\lim\ \inf_{y\to x} f(y)$ has a similar definition. It is easy to verify that the functions f^* and f_* are defined everywhere, and are upper semi-continuous (usc) and lower semi-continuous (lsc) respectively.

PROPOSITION. *If X is any topological space and $f\in L^\infty$, then* $2\ \text{dist}(f, C)\geqq d(f)\equiv\sup\{f^*(y)-f_*(y): y\in X\}$.

PROOF. If $f^*(x)-f_*(x)>d(f)-\epsilon$ and $g\in C$ then one or the other of $\lim\ \sup_{y\to x}(f(y)-g(y))$ and $\lim\ \sup_{y\to x}(g(y)-f(y))$ is greater than $\frac{1}{2}(d(f)-\epsilon)$.

THEOREM 1. *If X is paracompact, then $g\in C$ is a best approximator to $f\in L^\infty$ if, and only if, $f^*-\frac{1}{2}d(f)\leqq g\leqq f_*+\frac{1}{2}d(f)$; every $f\in L^\infty$ has such a best approximator; and* $\text{dist}(f, C)=1/2d(f)$.

PROOF. Since $f_*+\frac{1}{2}d(f)\geqq f^*-\frac{1}{2}d(f)$, the first pair of inequalities is equivalent to the condition that for every $\epsilon>0$ and every $x\in X$, there be a neighborhood U of x such that (ess sup$|f(y)-g(y)|: y\in U$) $\leqq\frac{1}{2}d(f)+\epsilon$. This in turn is equivalent to the assertion that for every $\epsilon>0$, $|f(y)-g(y)|>\frac{1}{2}d(f)+\epsilon$ only on a μ-null set, which says that $\|f-g\|\leqq\frac{1}{2}d(f)$. It remains only to show that there is a continuous function which satisfies these inequalities. Since $f^*-\frac{1}{2}d(f)$

is usc and $f_*+\frac{1}{2}d(f)$ is lsc, this follows from the Interposition Theorem of Dieudonné [1, p. 75].

THEOREM 2. *If X is a normal Hausdorff space then an element $f\in L^\infty$ has exactly one best approximator in C if, and only if, f^*-f_* is a constant function.*

PROOF. If f^*-f_* is constant, then the function $g=f^*-\frac{1}{2}d(f)=f_*+\frac{1}{2}d(f)$ is both lsc and usc, and hence is continuous. As in Theorem 1, $\|f-g\|=\text{dist}(f, C)$, and no other element of C has this property. Conversely, we must show that if f^*-f_* is not constant and f has a best approximator g in C, then it has more than one. If f^*-f_* is not constant, we can choose an $\epsilon>0$ and an $x\in X$ such that $f_*(x)+\frac{1}{2}d(f)-(f^*(x)-\frac{1}{2}d(f))=\epsilon$. Since g is continuous and f^* and f_* are semi-continuous, there is a neighborhood U of x on which $|g(y)-g(x)|<\epsilon/6$, $f_*(y)>f_*(x)-\epsilon/6$ and $f^*(y)<f^*(x)+\epsilon/6$. Since $\{x\}$ is closed and X is normal, Urysohn's Lemma asserts the existence of a non-negative function $p\in C$ such that $\|p\|=\epsilon/6$ and that p vanishes outside U. One or the other of the inequalities $f_*(x)+\frac{1}{2}d(f)-\epsilon/2\geqq g(x)$ and $g(x)\geqq f^*(x)-\frac{1}{2}d(f)+\epsilon/2$ must hold, so that either $f_*(y)+\frac{1}{2}d(f)-\epsilon/6>g(y)$ or $g(y)>f^*(y)-\frac{1}{2}d(f)+\epsilon/6$ on U. According to which is the case, put $h=g+p$ so that $g\leqq h\leqq f_*+\frac{1}{2}d(f)$ on U, or $h=g-p$ so that $f^*-\frac{1}{2}d(f)\leqq h\leqq g$ on U; $h=g$ on the complement of U. Then h is also a best approximator to f out of C.

If μ is the measure which assigns mass 1 to every point in X, then it certainly assigns positive mass to each nonempty open set, and mass 0 to each set which intersects a neighborhood of every point in a set of measure 0. In this case, L^∞ is the Banach space of all bounded functions on X, and $\|\ \|$ is the supremum norm. Theorems 1 and 2 thus solve, as a special case, the problem of approximating bounded functions by continuous functions in the uniform norm.

REFERENCE

1. J. Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures Appl. **23** (1944), 65–76.

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