# MULTIPLIERS IN $L^p$ AND INTERPOLATION

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For functions $u = u(\xi)$ in $\mathfrak{D}(R)^n$ ($C^\infty$ with compact support) we consider the mapping $K: u \to v$ defined by $\hat{v}(x) = k(x)\hat{u}(x)$. Here $k$ is a measurable function and "$\hat{\ }$" denotes the Fourier transform. If $K$ can be extended to a bounded map from $L^p$ to $L^s$ we say $k \in M_p^s$. ($M_p^p \equiv M_p$.) Elements in $M_p$ are called multipliers in $L^p$.

Hörmander [3] extended the result of Mikhlin [4] as follows:

THEOREM (HÖRMANDER). *A sufficient condition that $k(x)$ be a multiplier in $L^p$ for all $1 < p < \infty$ is that*

$$\text{(1)} \qquad \sup_{R>0} R^{-n} \int_{R<|x|<2R} |R^\alpha D^\alpha k|^2 \, dx < \infty$$

*for all $0 \leqq \alpha \leqq \nu$ and some $\nu > n/2$. (Derivatives in (1) are generalized derivatives.)*

On the other hand, it is well known that bounded measurable functions are multipliers in $L^2$. Hence it seems reasonable to interpolate between these two results to obtain a criterion for $k \in M_p$ for values of $1/p$ in a given range (which must be a symmetric interval about $1/2$, see [3].) That is the point of Theorem A.

THEOREM A. *If for some $q \geqq 2$*

$$\text{(2)} \qquad \sup_{R>0} R^{-n} \int_{R<|x|<2R} |R^\alpha D^\alpha k(x)|^q \, dx < \infty$$

*for all $0 \leqq \alpha \leqq \nu$ and some $\nu > n/q$, then $k \in M_p$ for all $p$ satisfying $|1/2 - 1/p| \leqq 1/q$. (If suitably interpreted $\alpha$ may be nonintegral; see below.)*

A slight improvement is

THEOREM B. *If for some $q \geqq 2$, $\beta > 0$*

$$\text{(3)} \qquad \sup_{R>0} R^{-n} \int_{R<|x|<2R} |R^{\alpha+\beta} D^\alpha k|^q \, dx < \infty$$

*for all $0 \leqq \alpha \leqq \nu$, some $\nu > n/q$, then*

$$k \in M_p^s \qquad \textit{if} \ \frac{1}{s} = \frac{1}{p} - \frac{\beta}{n} > 0.$$

NOTE. Theorem A has also been proved independently by Jaak Peetre in a paper to appear which, in addition, treats questions on the summability of Fourier integrals [personal communication].

We briefly sketch the proof of A. The method uses complex interpolation as explained, for instance, in [1], [2]. The interpolation space denoted in [2] by $[A, B, \delta(\theta)]$ we will denote by $[A, B]_\theta$, or simply by $[A, B]$. Let us further agree to denote the norm of an element $f$ in a Banach space $A$ by $A(f)$.

The proof of Theorem A is facilitated by the change of variable $t=\log|x|$, $\theta=x/|x|$. The mapping $T: x\to(\theta, t)$ maps $R^n-\{0\}$ onto the cylinder $C=S_{n-1}\times R_1$, and functions $f(x)$ are transplanted on $C$: $f(x)=f^*(\theta, t)$. We denote by $C_t$ the section of $C$: $\{\theta, \tau: |\tau-t|<1\}$ and by $H_{m,p,C_0}(k^*)$ the Sobolev norm of $k^*$ over $C_0$ (i.e., the $L^p$ norm over $C_0$ of $k^*$ and all derivatives of orders $\leqq m$. For nonintegral $m>0$ define this norm by interpolation. Set $H_{m,p,C_t}(k^*(\cdot, \cdot))\equiv H_{m,p,C_0}(k^*(\cdot, \cdot+t))$, and $\tilde{H}_{m,p}(k)\equiv\sup_{-\infty<t<\infty} H_{m,p,C_t}(k^*)$. Condition (1) is equivalent to

$$\tilde{H}_{\nu,2}(k) < \infty \qquad \text{for some } \nu > \frac{n}{2} \tag{4}$$

and condition (2) to

$$\tilde{H}_{\nu,q}(k) < \infty \quad \text{for some} \quad \nu > \frac{n}{q}\cdot \tag{5}$$

Thus Theorem A asserts

$$\textit{if (4) holds then } k \in M_p \textit{ for } \left|\frac{1}{2}-\frac{1}{p}\right| \leqq \frac{1}{q}\cdot \tag{6}$$

We now state a series of lemmas used in the proof.

LEMMA 1. *If* (4) *holds for some real* $\nu>n/2$ *then* $k\in M_p$ *for all* $1<p<\infty$. (*This follows by slightly modifying Hörmander's proof.*)

LEMMA 2.

$$[H_{m_1p_1}, H_{m_2p_2}]_\theta = H_{mp}$$

*where* $m=(1-\theta)m_1+\theta m_2$, $1/p=(1-\theta)/p_1+\theta/p_2$ *and* $1<p_1, p_2<\infty$.

LEMMA 3. *Same with* $\tilde{H}$ *spaces.*

LEMMA 4. *Suppose* $A_1$, $B_1$, $A_2$, $B_2$ *are Banach spaces and* $A_1\subset B_1$, $A_2\subset B_2$ *with corresponding inequalities for the norms, then*

$$[A_1, A_2] \subset [B_1, B_2]$$

*with the corresponding inequalities for norms.*

LEMMA 5. $[M_{p_1}, M_{p_2}] \subset M_p$, $1/p = (1-\theta)/p_1 + \theta/p_2$, *with corresponding inequalities for norms.*

LEMMA 6. *It suffices to prove Theorem* A *for all* $p$ *in the open interval* $|1/2 - 1/p| < 1/q$.

PROOF OF THEOREM A. We take $\theta = 2/q$, then for sufficiently small $\epsilon > 0$, we have, for $|1/2 - 1/p| < 1/q$,

$$M_p \supset [M_2, M_{1/\epsilon}] \supset [\tilde{H}_{2\epsilon, n/\epsilon}, \tilde{H}_{n/2+\epsilon, 2}] = \tilde{H}_{\alpha,\beta} \supset \tilde{H}_{\nu,q}$$

where $\alpha = 2\epsilon(1-2/q) + 2(n/2+\epsilon)/q$, $1/\beta = \epsilon(1-2/q)/n + 1/q$ and $\nu > n/q$.

Theorem B follows from Theorem A by considering the operator $K_1$ with multiplier $k_1 \equiv r^{-\beta} k$.

It may seem of little use to know Theorem A for fractional $\alpha$. However we have the following consequence of Theorem A which does not involve fractional derivatives.

THEOREM C. *Suppose* $k$ *is bounded and sufficiently smooth for* $x \neq 0$ *(for the following quantities to make sense) and for some* $0 < \theta < 1$ *the quantity*

$$\left[R^{-n} \int_{R<|x|<2R} |R^{m_1} D^{m_1} k|^{q_1} dx\right]^{(1-\theta)/q_1} \cdot \left[R^{-n} \int_{R<|x|<2R} |R^{m_2} D^{m_2} k|^{q_2} dx\right]^{\theta/q_2}$$

*is bounded for* $R > 0$. *Assume further that*

$$(1-\theta)\left(m_1 - \frac{n}{q_1}\right) + \theta\left(m_2 - \frac{n}{q_2}\right) > 0.$$

*Then* $k \in M_p$ *for* $|1/2 - 1/p| \leq (1-\theta)/q_1 + \theta/q_2$. *(In the above,* $D^m$ *denotes a generic derivative of order* $m$.)

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