RESEARCH PROBLEMS

12. Richard Bellman: *Matrix theory*

All matrices that appear are $N \times N$ symmetric matrices, $N \geqq 2$, and $A \geqq B$ signifies that $A - B$ is nonnegative definite. Consider the set of matrices X with the property that $X \geqq A_1$ and $X \geqq A_2$ where A_1 and A_2 are two given matrices. Let us choose an element in this set (or the element) with the property that $g(X)$, a prescribed scalar function of X, is minimized. For example, $g(X)$ might be $\operatorname{tr}(X)$, $\operatorname{tr}(X^2)$, or the largest characteristic root of X.

This procedure defines a function of A_1 and A_2 which we denote by $m(A_1, A_2)$. Similarly, we may define $m(A_1, A_2, A_3)$, and, generally, $m(A_1, A_2, \cdots, A_k)$ for any $k \geqq 2$.

Can we find a function $g(X)$ with the property that $m(A_1, A_2, A_3) = m(A_1, m(A_2, A_3))$? If so, determine all such functions, and for general $k \geqq 3$ as well. (Received April 10, 1965.)

13. Richard Bellman: *Differential approximation*

Let $y(t)$ be a given vector function belonging to $L^2(0, T)$ and let $x(t)$ be determined as the solution of the linear vector differential equation $x' = Ax$, $x(0) = c$. Under what conditions on c and y does the expression $\int_0^T (x-y, x-y)\, dt$ possess a minimum rather than an infimum with respect to the constant matrix A? (Received May 12, 1965.)

14. Richard Bellman: *A limit theorem*

It is well known that if $u_n \geqq 0$ and $u_{m+n} \leqq u_m + u_n$, for $m, n = 0, 1, \cdots$, then $u_n \sim nc$ as $n \rightarrow \infty$ for some constant c. Let $u_n(p)$ be a function of p for $p \in S$, a given set, and $T(p)$ be a transformation with the property that $T(p) \in S$ whenever $p \in S$, $i = 1,2$. Suppose that

$$u_{m+n}(p) \leqq u_m T_1(p) + u_n(T_2(p))$$

for all $p \in S$ and $m, n = 0, 1, \cdots$. Under what conditions on $T_1(p)$ and $T_2(p)$ and S is it true that $u_n(p) \sim ng(p)$ as $n \rightarrow \infty$? When is $g(p)$ independent of p? (Received May 12, 1965.)

15. Richard Bellman: *Generalized existence and uniqueness theorems*

Given a second-order linear differential equation $u'' + p(t)u' + q(t)u = 0$, subject to various initial and boundary conditions, there are two types of problems we can consider. The first are the classical existence and uniqueness theorems; the second are often called "inverse problems," where the problem is that of determining the properties of the coefficients from the properties of the solution. For a version of the second type, of importance in modern physics, see, for example, B. M. Levitan and M. G. Gasymov, *Determination of a differential equation by two of its spectra*, Russian Math. Surveys **19**, No. 2 (1964), 1–64. References to earlier work by Borg, Levinson, and others will be found there.

Let us now consider a class of problems containing both of the foregoing as special cases. A simple version is the following. Suppose that $0 \leqq t \leqq T$, and that S_1, S_2, S_3, S_4 are subsets of the interval $[0, T]$. What classes of sets S_1, S_2, S_3, S_4, and what types of conditions on u in S_1, u' in S_2, p in S_3, and q in S_4, determine u, p, and q in $[0, T]$?

Analogous problems of greater complexity and generality can be posed for higher-order and nonlinear differential equations, for partial differential equations, and for all of the classes of functional equations of analysis. In abstract form, we consider a functional equation $u = T(u, v, a)$, where u, v are functions and a is a vector parameter. Partial information is given concerning u, v, and a, and it is required to deduce all of the missing information. There are analogous problems for variational and control processes. (Received May 21, 1965.)

16. L. Carlitz: *A Saalschützian theorem for double series*

Saalschütz proved that

$$ {}_3F_2\left[\begin{matrix} -m, & a, & b; \\ & c, & d \end{matrix}\right] = \frac{(c-a)_m(c-b)_m}{(c)_m(c-a-b)_m}, $$

provided

$$ c + d = a + b - m + 1. $$

The writer (J. London Math. Soc. **38** (1963), 415–418) proved that the series

$$ S = \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-m)_r(-n)_s(a)_{r+s}(b)_r(b')_s}{r!s!(c)_{r+s}(d)_r(d')_s} $$

satisfies

$$ S = \frac{(b+b'-a)_{m+n}(b')_m(b)_n}{(b+b')_{m+n}(b'-a)_m(b-a)_n} $$

provided

$$c + d = a + b - m + 1,$$
$$c + d' = a + b' - n + 1,$$
$$c = b + b'.$$

Now these conditions imply

$$(*) \qquad 2c + d + d' = 2a + b + b' - m - n + 2.$$

The question arises whether S can be summed when only the condition (*) is assumed. (Received May 15, 1965.)

17. Olga Taussky: *Matrix theory*

A. It is known that the $n \times n$ hermitian matrices are closed under the Jordan product $AB+BA$. This composition is commutative. A noncommutative composition generalizing the Jordan product can be given for general complex matrices (or matrices over an abstract field with an involution) in the following way

$$AB + B^*A^*.$$

By X^* is meant the complex conjugate and transposed matrix. Study the structure of this generalized Jordan algebra. The idea to study this composition comes from Lyapunov's theorem which states that a matrix whose characteristic roots have positive real parts has a unique "generalized Jordan product" hermitian inverse which is a positive definite hermitian matrix (see O. Taussky, *A remark on a theorem of Lyapunov*, J. Math. Anal. Appl. **2** (1961), 105–107).

B. Let A be an $n \times n$ matrix with complex elements and characteristic roots with negative real parts. It is known (theorem of Lyapunov) that such matrices are characterized by the fact that a positive definite matrix G exists with

$$AG + GA^* \text{ negative definite.}$$

What is the range of $AG+GA^*$ if G runs through all positive definite $n \times n$ matrices? A^* is the complex conjugate and transposed matrix.

C. What can one say about pairs of matrices which can be transformed to Jordan normal form simultaneously by a similarity? (Received May 20, 1965.)