HERBRAND ANALYZING FUNCTIONS

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In [1] it was shown that there is no strong analyzing function for crucial applications of Herbrand's rules of passage, and, hence, that there is no such function for the rule of modus ponens. However, there are weak analyzing functions for these rules, and in parts (a) and (b) of the following theorem two simple ones are specified.

THEOREM. *Let S and T be any closed quantificational schemata containing j and k quantifiers respectively. Let $r=j+k$, and let $m=\max(p, q)$, where $j, k, p, q \geqq 1$. Finally, let $\gamma(j, p)$ be the function*

$$j^{j^{p}},$$

and let $\phi(j, k, p, q)$ be the function

$$\left.r^{r^{\cdot^{\cdot^{\cdot^{r^m}}}}}\right\} 2j^2.$$

(a) *If S has property C of order p, and T comes from S by one crucial application of a rule of passage, then T has property C of order $\gamma(j, p)$.*

(b) *If S has property C of order p, and the schema $S \supset T$ has property C of order q, then T has property C of order $\phi(j, k, p, q)$.*

(c) *There is neither a 3-placed function $\delta(j, p, q)$ nor a 3-placed function $\zeta(k, p, q)$ such that whenever S has property C of order p and $S \supset T$ has property C of order q, then T has property C either of order $\delta(j, p, q)$ or of order $\zeta(k, p, q)$.*

The functions γ and ϕ do not give the least possible bounds, but they do make clear that the only information needed about the schemata—in addition to property C orders—is the number of quantifiers occurring in the schemata. The argument showing that these functions γ and ϕ are weak analyzing functions will appear in Dreben's introduction to [2]. It turns on the formula stated at the end of [1]. Here we shall prove part (c) of the theorem by means of examples.

EXAMPLE 1. Let S be the schema

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$$(y_0)[(y_1)\neg My_0y_1 \vee (Ex_0)My_0x_0];$$

for each $t \geqq 3$, let A_t be the schema

$$(y_2) \cdots (y_t)(Ex_2)[(Gy_2x_2 \,\&\, \neg My_3x_2) \vee (Gy_3x_2 \,\&\, \neg My_4x_2) \vee \cdots \vee (Gy_{t-1}x_2 \,\&\, \neg My_tx_2) \vee \neg My_2y_2 \vee Gy_tx_2];$$

and let T_t be the schema

$$A_t \vee (Ex_1)(Ex)(Mx_1x \,\&\, (y)\neg Gx_1y).$$

Now S contains three quantifiers. Moreover, S has property C of order 2, and $S \supset T_t$ has property C of order 4. But for each $t \geqq 3$, the schema T_t has property C of no order earlier than $t+1$. Hence, no function satisfying the conditions on δ can exist.

EXAMPLE 2. For each $s \geqq 2$, let S_s be the schema

$$[(y_1)\neg M_1y_1 \vee (Ex_1)M_1x_1] \,\&\, [(y_2)\neg M_2y_2 \vee (Ex_2)M_2x_2] \,\&\, \cdots \,\&\, [(y_s)\neg M_sy_s \vee (Ex_s)M_sx_s]$$

and let T_s be the schema

$$(Ex)(y)[\neg M_1y \vee M_sx \vee (M_1x \,\&\, \neg M_2y) \vee (M_2x \,\&\, \neg M_3y) \vee \cdots \vee (M_{s-1}x \,\&\, \neg M_sy)].$$

The schema T_s contains just two quantifiers, but has property C of no order earlier than $s+1$. However, the schema S_s has property C of order 2, and the schema $S_s \supset T_s$ has property C of order 3. So there is no function ζ.

REFERENCES

1. Burton Dreben, Peter Andrews and Stål Aanderaa, *False lemmas in Herbrand*, Bull. Amer. Math. Soc. **69** (1963), 699–706.

2. John van Heijenoort, Editor, *Jacques Herbrand, Ecrits logiques*, Presses Universitaires, Paris.

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