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A NOTE ON APPROXIMATION BY BERNSTEIN POLYNOMIALS

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Let f be continuous on $[0, 1]$ and $0 \leqq \alpha < \beta \leqq 1$ and let $B_n f$ be the Bernstein polynomial of f of degree n, defined by

$$B_n f(x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{n}{\nu} x^\nu (1 - x)^{n-\nu}.$$

In view of a result of E. V. Voronovskaya, which states that the boundedness of f on $[0, 1]$ and the existence of f'' at a point $x \in [0, 1]$ implies that

$$B_n f(x) - f(x) = \frac{x(1 - x)}{2n} f''(x) + o\left(\frac{1}{n}\right) \qquad (n \to \infty),$$

it has been conjectured [1, p. 22] that the relation

$$B_n f(x) - f(x) = o\left(\frac{1}{n}\right)$$

cannot be true for all $x \in [\alpha, \beta]$ unless f is a linear function on $[\alpha, \beta]$. The following theorem related to this conjecture was proved by K. de Leeuw [2]:

If f is continuous on $[0, 1]$ and

$$B_n f(x) - f(x) = O\left(\frac{1}{n}\right)$$

holds uniformly on every subinterval $[\alpha, \beta]$ of $[0, 1]$ and if in addition

$$B_nf(x) - f(x) = o\left(\frac{1}{n}\right)$$

at almost all points of $[\alpha, \beta]$, then f is linear on $[\alpha, \beta]$.

We shall give here a simple proof of the original conjecture.

THEOREM. *If f is continuous on $[0, 1]$ and*

$$B_nf(x) - f(x) = o\left(\frac{1}{n}\right) \qquad (n \to \infty) \tag{1}$$

holds for each fixed $x \in (\alpha, \beta)$, then f is a linear function on $[\alpha, \beta]$.

PROOF. To make the argument as transparent as possible we list first the properties of Bernstein polynomials used in this proof.

(i) $B_n(f+g) = B_nf + B_ng$ and if $f \leqq g$ on $[0, 1]$ then $B_nf \leqq B_ng$.

(ii) If h is a linear function on $[0, 1]$, then $B_nh = h$.

(iii) If $Q(x) = Ax^2 + Bx + C$, then $B_nQ(x) - Q(x) = A(x(1-x)/n)$.

(iv) If g is bounded on $[0, 1]$ and $g = 0$ on $[\alpha, \beta]$, then $B_ng(x) = o(1/n)$ $(n \to \infty)$ for each fixed x *in the interior* of $[\alpha, \beta]$. (Actually, $B_nf(x) = O(e^{-\delta(x)n})$ $(n \to \infty)$, with $\delta(x) > 0$, but the weaker property is sufficient.)

We need also the following lemma:

LEMMA. *If f is continuous on $[\alpha, \beta]$, vanishes at α and β and has a positive maximum on $[\alpha, \beta]$ then there is a quadratic polynomial $Q(x) = Ax^2 + Bx + C$ with $A < 0$ such that*

$$f(x) \leqq Q(x) \quad \textit{for all } x \in [\alpha, \beta] \tag{2}$$

and

$$f(c) = Q(c) \quad \textit{for some } c \textit{ in the interior of } [\alpha, \beta]. \tag{3}$$

This lemma is geometrically almost obvious. We can namely choose the parabola $P(x) = Ax^2 + Bx + C^*$ with $A < 0$ such that its arc over $[\alpha, \beta]$ lies in the strip $M \leqq y \leqq 3M/2$ where $M = \max_{\alpha \leqq x \leqq \beta} f(x) > 0$. If $d = \min_{\alpha \leqq x \leqq \beta} (P(x) - f(x))$, then the quadratic polynomial $Q(x) = P(x) - d$ has the required properties.

To prove the theorem, suppose that f satisfies (1) for each $x \in [\alpha, \beta]$. By subtracting a suitable linear function and using (ii), if necessary, we may assume that $f(\alpha) = f(\beta) = 0$. We have to show that $f = 0$ on $[\alpha, \beta]$.

Assume that the maximum of f on $[\alpha, \beta]$ is positive. Then by the preceding lemma we can find a polynomial Q satisfying (2) and (3). Since by (2) $f(x) \leqq Q(x)$, $x \in [\alpha, \beta]$, we can find a bounded function g on $[0, 1]$ such that $g=0$ on $[\alpha, \beta]$ and

$$f(x) \leqq Q(x) + g(x) \quad \text{for all } x \in [0, 1].$$

Applying (i) we get

$$B_n f(x) \leqq B_n Q(x) + B_n g(x).$$

Putting here $x=c$ and using (3) we get

$$B_n f(c) - f(c) \leqq B_n Q(c) - Q(c) + B_n g(c).$$

Since c is *in the interior* of $[\alpha, \beta]$ we have by (iv) $B_n g(c) = o(1/n)$ $(n \to \infty)$. Using this result and (iii) we obtain

$$B_n f(c) - f(c) \leqq A \frac{c(1-c)}{n} + o\left(\frac{1}{n}\right) \qquad (n \to \infty),$$

with $A < 0$, which is impossible by (1). Thus the maximum of f on $[\alpha, \beta]$ cannot be positive.

Likewise, by considering $-f$ instead of f, we see that the minimum of f on $[\alpha, \beta]$ cannot be negative.

Thus, $f=0$ on $[\alpha, \beta]$, and the theorem is proved.

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