ON THE THICKNESS OF THE COMPLETE GRAPH[1]

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The *thickness* $t(K_p)$ of the complete graph K_p with p points is the minimum number of planar subgraphs whose union is K_p. The purpose of this note is to outline a result which determines $t(K_p)$ for four of every six consecutive integers p. A complete proof of this result will be published elsewhere.

THEOREM. *If* $p \equiv -1, 0, 1, 2 \pmod 6$, *then*

$$t(K_p) = \left[\frac{p+7}{6}\right]. \tag{1}$$

In proving this theorem, we prescribe a labelling of $n+1$ plane graphs, for any positive integer n. All the graphs contain the same $6n+2$ points, but are constructed so that no two have a common line. Two of the points will be denoted by v and v', and the others as $u_k, v_k, w_k, u_k', v_k', w_k'$ for $k=0, 1, \cdots, n-1$. All but one of the graphs are of the type indicated in Figure 1, where each of the six numbered triangles in G_k contains $n-1$ other points and $3(n-1)$ lines so that its interior is isomorphic with graph H.

The points of the n graphs G_k are labelled using an $n \times n$ matrix $A=(a_{ij})$, whose entries are residue classes modulo n, where

$$a_{ij} = \left((-1)^i\left[\frac{i}{2}\right] + (-1)^j\left[\frac{j}{2}\right]\right) \pmod n \tag{2}$$

with $[x]$ indicating the greatest integer function as usual. We remark that one of the important properties of A is that each residue class appears exactly once in each row and each column.

The $n-1$ points inside triangle $u_k' v_k w_k'$ of graph G_k are labelled using the column, say the jth, whose first entry is $a_{1j}=k$ as follows: if $a_{ij}=h$, the $(i-1)$st point down from v_k is labelled v_h or v_h' according as $\min\{i, j\}$ is odd or even. The points inside triangle $v_k u_k' w_k$ are similarly labelled, using u_h' and u_h instead of v_h and v_h' respectively. The points inside the other triangles are also labelled analogously.

Now, in the union of these n labelled graphs G_k, aside from v and v', each point is adjacent with all but one of the other points. More-

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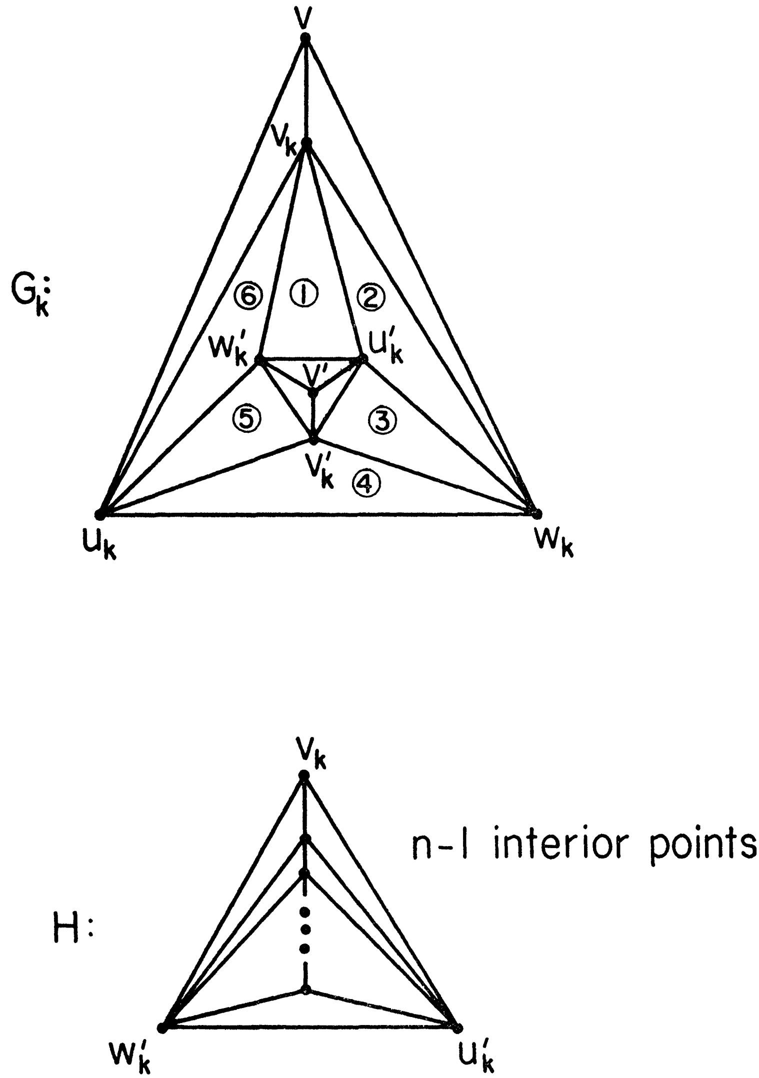

FIGURE 1

over, for each integer k, the points u_k, v_k, and w_k are not adjacent to u_k', v_k', and w_k', respectively. Also, v and v' are each adjacent with half of the other points. A new graph G is constructed as in Figure 2, in which each of the $6n+2$ points is adjacent to all of the points not adjacent to it in any of the other n graphs. Therefore the union of

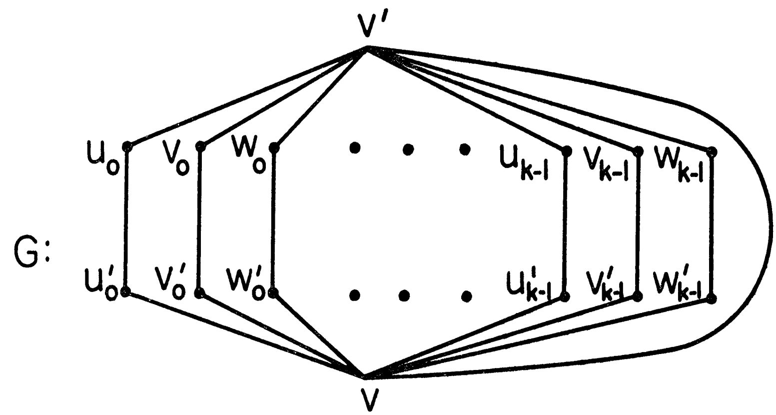

FIGURE 2

the graph G with the n graphs G_k is complete. Thus, $t(K_{6n+2}) \leqq n+1$. From Euler's polyhedron formula it follows that $t(K_{6n-1}) \geqq n+1$. The theorem follows at once from these two inequalities.

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