PERIODIC MAPS WHICH PRESERVE A COMPLEX STRUCTURE

BY P. E. CONNER AND E. E. FLOYD

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1. **Introduction.** A *weakly complex structure* for a differentiable manifold M is, roughly, a structure for the stable tangent bundle of M as a complex vector space bundle; a map $T: M \rightarrow M$ is *weakly complex* if the differential dT is stably complex linear. We consider weakly complex maps $T: M \rightarrow M$, periodic of period p (usually p is prime). We study such problems as the relationship between M, F and the normal bundle to F.

There are two basic technical tools. First we study the complex bordism groups $\mathfrak{U}_n(X)$ of a space X, as a generalized homology theory. For B_{Z_p} a classifying space for the group Z_p, $\mathfrak{U}_n(B_{Z_p})$ is identified with the bordism group of weakly complex maps $T: M \rightarrow M$ of prime period p, operating on a closed manifold M without fixed points. The second technical tool is the theory of G-bundles $E \rightarrow B$ where a compact Lie group H acts on E as a group of bundle maps.

This work is a supplement to our previous study of periodic maps [1], and the methods are a continuation of those. A sample of the results here were given in our Seattle lectures [2]; Zelle has also studied aspects of weakly complex actions in his thesis [6]. A full account of our results will appear later.

2. **The complex bordism groups.** Given a bundle ξ of real $2k$-planes over a space X, a complex prestructure for ξ is a map J mapping each fiber of ξ linearly into itself and having $J^2 = -1$. A *complex structure* for ξ is a homotopy class of such prestructures; denote by $C(\xi)$ the set of complex structures. Denote by kI the trivial k-plane bundle $R^k \times X \rightarrow X$. For X a finite dimensional CW complex and for ξ a bundle of real n-planes over X, a *weakly complex structure* for ξ is an element of $C((2k-n)I+\xi)$, $2k-2 \geqq \dim X$; in an appropriate sense, this is independent of k.

A *weakly complex manifold* is a pair consisting of a differentiable manifold M and a weakly complex structure on the tangent bundle of M. The boundary of a weakly complex manifold is weakly complex; each weakly complex manifold has a uniquely defined negative.

Given a pair (X, A) of spaces, consider all pairs (M, f) where M is a weakly complex compact n-manifold and where $f: (M, \partial M) \rightarrow (X, A)$. Two such, (M_1, f_1) and (M_2, f_2), are bordant if there exists a weakly complex compact $(n+1)$-manifold W and a map $F: W \to X$ such that

(i) disjoint copies of M_1 and $-M_2$ are contained as smooth submanifolds of ∂W with weakly complex structure induced from that of ∂W,

(ii) $F|M_i = f_i$ and $F(\partial W - (M_1 \cup M_2)) \subset A$. Bordism is an equivalence relation; denote the equivalence class represented by (M, f) by $[M, f]_U$ or simply by $[M, f]$. The set $\mathfrak{U}_n(X, A)$ of all equivalence classes is the *complex bordism group* of (X, A); in Milnor's notation this would be $\Omega_n^U(X, A)$, and Atiyah's notation would be $MU_n(X, A)$. There is $\partial: \mathfrak{U}_n(X, A) \to \mathfrak{U}_{n-1}(A)$ given by $\partial[M, f] = [\partial M, f|\partial M]$. If $\phi: (X, A) \to (Y, B)$, $\phi_*: \mathfrak{U}_n(X, A) \to \mathfrak{U}_n(Y, B)$ is given by $\phi_*[M, f] = [M, \phi f]$. Let $\mathfrak{U}_*(X, A) = \sum \mathfrak{U}_n(X, A)$.

$\{\mathfrak{U}_*(X, A), \partial, \phi_*\}$ *is a generalized homology theory*; that is, it satisfies the Eilenberg-Steenrod axioms except for the dimensional axiom. The coefficient group $\mathfrak{U}_*(\text{point})$ is the Milnor bordism ring $\mathfrak{U}_*$ (in his notation, Ω_*^U), where $\mathfrak{U}_n$ consists of all bordism classes $[M]$ of closed weakly complex n-manifolds. $\mathfrak{U}_*$ is a polynomial algebra over Z with a generator in each positive dimension $2k$.

There is the Milnor spectrum

$$MU: \cdots, MU(k), SMU(k), MU(k+1), \cdots$$

where $MU(k)$ is the Thom space of the universal $U(k)$-bundle, and where the map $S^2MU(k) \to MU(k+1)$ is given by Milnor [**4**]. In the following, the homology theory of a spectrum is due to G. W. Whitehead [**5**].

(2.1) *On the category of* CW *pairs, the complex bordism homology theory is isomorphic to the homology theory of the spectrum MU; we have*

$$\mathfrak{U}_n(X, A) \cong H_n(X, A; MU) = \pi_{n+2k}(MU(k) \wedge (X/A)),$$

k large.

There is a spectral sequence $\{E_{p,q}^r\}$ with $E_{p,q}^2 = H_p(X, A; \mathfrak{U}_q)$ and whose E^∞-term is associated with a filtration of $\mathfrak{U}_*(X, A)$. There is also a homomorphism $\mu: \mathfrak{U}_n(X, A) \to H_n(X, A)$ given by $\mu[M, f] = f_*(\sigma)$ where $\sigma \in H_n(M, \partial M)$ is the orientation class (a weakly complex manifold has a natural orientation).

(2.2) *For (X, A) a* CW *pair, the spectral sequence associated with $\mathfrak{U}_*(X, A)$ is trivial iff $\mu: \mathfrak{U}_*(X, A) \to H_*(X, A)$ is an epimorphism. If $H_*(X, A)$ has no torsion, then μ is an epimorphism. Given a set $\{[M_i, f_i]\}$ of homogeneous elements of $\mathfrak{U}_*(X, A)$ such that $\{\mu[M_i, f_i]\}$ is a basis for $H_*(X, A)$, then $\mathfrak{U}_*(X, A)$ is a free $\mathfrak{U}_*$-module with base $\{[M_i, f_i]\}$.*

3. Preliminaries on weakly complex actions. Let the compact Lie group H act differentiably on the compact differentiable manifold M; then H also acts on the tangent bundle $\xi: E \to M$ via the differentials dh, $h \in H$. There is a Riemannian metric on M invariant under the action of H; the exponential map $\exp: E \to M$ is then equivariant. If W is a compact smooth submanifold of M, invariant under H, then exp maps a normal cell bundle E_ϵ diffeomorphically and equivariantly onto a tubular neighborhood N of W in M. Hence we may identify N with the cell bundle, and the action becomes an action by bundle maps. If $F = F(H, M)$ is the set of *stationary* points of the action, then F is a finite disjoint union of smooth submanifolds of M; we treat it as if it were a manifold. The tubular neighborhood N of F is of the above type.

If $\xi: E \to M$ is the tangent bundle to M, then H acts on the Whitney sum $(2k-n)I + \xi$ as a group of bundle maps, acting trivially on the first coordinate. An *invariant* complex prestructure J on $(2k-n)I+\xi$ is a prestructure which commutes with the action of H. An *invariant complex structure* is a homotopy class of such prestructures.

A weakly complex action of the compact Lie group H on the differentiable manifold M is a pair consisting of a differentiable action of H on M and an invariant weakly complex structure for the action. We regard the stable tangent bundle $(2k-n)I+\xi$ as a complex vector space bundle, and the differentials dh as complex linear.

(3.1) *Consider a weakly complex action of H on M, where J is the invariant prestructure on the stable tangent bundle $(2k-n)I+\xi$. The restriction $(2k-n)I+\xi_F$ to the stationary point set F splits into the stable tangent bundle $(2k-n)I+\xi'$ to F and the normal bundle η to F in M, and each is invariant under J. Hence F is a weakly complex manifold and the normal bundle η to F is a unitary bundle, with H acting on η as a group of complex linear bundle maps.*

Consider now a tubular neighborhood N of the set F of stationary points of a weakly complex action. On the one hand, N receives an invariant weakly complex structure by restriction of that of M. On the other hand, F is weakly complex and the normal bundle η to F is unitary. The stable tangent bundle ξ of N splits as the sum of the stable normal bundle ξ' to the fiber and the tangent bundle η' along the fiber. Now ξ' receives a weakly complex structure from F, and η' from the unitary structure of η. Hence N receives a second invariant weakly complex structure. *These two invariant weakly complex structures coincide. Hence the weakly complex action of H on N is determined completely by the weakly complex manifold F, the normal bundle η to F with its unitary structure, and the action of H on η.*

4. **Weakly complex maps of prime period.** Consider a free, weakly complex action of a finite group H on a closed n-manifold M; denote the pair by (H, M). There is a natural equivariant bordism group of such pairs; denote the bordism class represented by (H, M) by $[H, M]_U$ or simply by $[H, M]$. Given a pair (H, M), then M/H is weakly complex and $M \to M/H$ is a principal H-bundle. Letting $f: M/H \to B_H$ be a classifying map for the principal H-bundle, we receive an element $[M/H, f] \in \mathfrak{U}_n(B_H)$. There results an isomorphism of the bordism group of free weakly complex actions with $\mathfrak{U}_n(B_H)$; we identify the two. The $\mathfrak{U}_*$-module structure of $\mathfrak{U}_*(B_H)$ is given by letting $[H, M][M']$ denote $[H, M \times M']$ where H acts on $M \times M'$ by $h(x, y) = (hx, y)$.

In the remainder of this section consider $H = Z_p$, p a prime. An action of Z_p is equivalent to a map $T: M \to M$ of period p; we use either $[Z_p, M]$ or $[T, M]$ for the element of $\mathfrak{U}_*(B_{Z_p})$.

Consider first a weakly complex map $T: M \to M$, on a closed manifold, of period p, where T has fixed points. If B is the boundary of a tubular neighborhood N of the fixed point set F, then T on B gives a free action of Z_p and $[T, B] = 0$ in $\mathfrak{U}_*(B_{Z_p})$. In this way we get relations in $\mathfrak{U}_*(B_{Z_p})$.

Let η be the normal bundle to F and let η' be the Whitney sum of a trivial complex line bundle and η. An action of Z_p on the line bundle is given by multiplication by $\exp 2\pi i/p$; the diagonal action gives an action of Z_p on η'. Let B' be the sphere bundle of η', and let $T': B' \to B'$ be the periodic map resulting from the action of Z_p on η'.

(4.1) *If $T: M \to M$ is a weakly complex map of prime period p and if B' is as above, then $[T', B'] = [T_1, S^1][M]$ in $\mathfrak{U}_{n+1}(B_{Z_p})$, where $T_1(z) = (\exp 2\pi i/p) \cdot z$.*

The proof is similar to [1, Theorem 35.1]. We can go on to analyze $\mathfrak{U}_*(B_{Z_p})$ as in [1]. For each S^{2n-1}, pick a unitary $T: S^{2n-1} \to S^{2n-1}$ of period p and without fixed points. Then $\{[T, S^{2n-1}]\}$ generates the reduced group $\tilde{\mathfrak{U}}_*(B_{Z_p})$ as a $\mathfrak{U}_*$-module. Moreover $[T, S^{2n-1}]$ is of order p^{k+1} where $2k(p-1) < 2n-1 < 2(k+1)(p-1)$. $[T, S^{2k(p-1)+1}]$ is of particular interest. Here $p^k[T, S^{2k(p-1)+1}] = b[T_1, S^1][P_{p-1}(C)]^k$, where $b \not\equiv 0 \bmod p$. The explicit additive structure of $\mathfrak{U}_*(B_{Z_p})$ is obtained.

Perhaps the most interesting feature in the complex case is the case $p = 2$, which is completely different from the oriented bordism situation. Consider $\mathfrak{U}_*(B_{Z_2}) = \mathfrak{U}_*(P_\infty(R))$, or alternatively the bordism theory of fixed point free weakly complex involutions. The generators are $[T, S^{2n-1}]$ where T is the antipodal map, $[T, S^{2n-1}]$ is of order 2^n and $2^{n-1}[T, S^{2n-1}] = [T_1, S^1][P_1(C)]^{n-1}$.

(4.2) *Suppose that M^{2n} is a closed almost complex manifold and that $T: M^{2n}\to M^{2n}$ is a differentiable involution commuting with the almost complex structure. If T has only isolated fixed points, then the number of fixed points is of the form $k\cdot 2^n$. Moreover $[M^{2n}]=k[P_1(C)]^n$ in $\mathfrak{U}_*/2\mathfrak{U}_*$.*

As an example it is easy to construct a complex analytic involution on $[P_1(C)]^n$ having exactly 2^n fixed points.

5. **Weakly complex actions of $(Z_p)^k$ and Z_{p^k}.** Consider a weakly complex action of $(Z_p)^k$, p a prime; that is, consider maps $T_i: M\to M$, $i=1, \cdots, k$, of period p which commute and all of which have invariant the same weakly complex structure on M. Our previous methods [1, §43] apply to show that if $(Z_p)^k$ acts in a weakly complex fashion on the closed n-manifold M without stationary points, then the Chern numbers of M are all divisible by p. We strengthen this to the following.

(5.1) *The ideal of $\mathfrak{U}_*$ consisting of all elements admitting a representative M upon which there is a weakly complex action of $(Z_p)^k$ without stationary points coincides with the ideal of $\mathfrak{U}_*$ consisting of all elements whose Chern numbers are all divisible by p.*

In the above, there is exhibited a sequence $p, M^{2p-2}, \cdots, M^{2p^k-2}, \cdots$ with $(Z_p)^{k+1}$ acting as required on M^{2p^k-2}. Here M^{2p^k-2} is the submanifold of $P_{p^k}(C)$ consisting of all $[z_0, \cdots, z_{p^k}]$ with $\sum z_i^p=0$. Each generator of $(Z_p)^k$ sends $[z_0, \cdots, z_i, \cdots]$ into $[\rho_0 z_0, \cdots, \rho_i z_i, \cdots]$ for appropriate choice of pth roots ρ_i of unity. The bordism classes $p, [M^{2p-2}], [M^{2p^2-2}], \cdots$ generate the ideal of elements of $\mathfrak{U}_*$ all of whose Chern numbers are divisible by p.

In a similar fashion, we can consider the ideal of the oriented bordism ring Ω_* represented by manifolds M upon which some $(Z_p)^k$ acts, preserving the orientation and without stationary points, for p an odd prime. This ideal consists of all elements having all Pontryagin numbers divisible by p. The analogous problem in which k is fixed continues to be unsolved.

We consider now actions of Z_{p^k}, solving a problem that was only treated with partial success in our previous work [1, §45].

(5.2) *Suppose that $T: M\to M$ is a weakly complex map of prime power period p^k on the closed manifold M. If T has no fixed points, then $[M]\in p\mathfrak{U}_*$. Similarly if T is a differentiable orientation preserving map of prime power period p^k, p an odd prime, acting on the closed oriented manifold without fixed points, then $[M]\in p\Omega_*$.*

Besides using the results of §4, the proof uses a desingularization process based on expressing every lens space as a boundary in a specific way. Namely consider a closed manifold M upon which the compact Lie group G acts in a weakly complex fashion. Let $S^1\subset G$ be in the center of G and suppose S^1 has no stationary points in M. Then there exists a weakly complex action of G on a compact manifold W so that $\partial W = M$ and so that the action of G on W restricts to the action of G on M.

For $p=2$, the conclusion of (5.2) in the oriented case would be definitely false [1, §45]. However for p odd the normal bundle η to the fixed point set F can be reduced to the unitary group, and the methods of the complex case apply.

6. **Equivariant maps.** In this section we indicate the application of bordism to nonexistence theorems for equivariant maps. For simplicity of statement, we confine ourselves to $p=2$; the results can be extended to any p.

(6.1) THEOREM. *Consider an element $\gamma \in \tilde{\mathfrak{U}}_{2n-1}(B_{Z_2})$. The order of γ divides 2^k if and only if there exists a representative (T, M^{2n-1}) of γ and a map $f: M^{2n-1} \to S^{2k}$ with $f(Tx) = -f(x)$ for all $x \in N^{2n-1}$.*

In a different terminology, the order of γ divides 2^k if and only if some representative (T, M^{2n-1}) is of co-index $\leqq 2k$. Hence it becomes of geometric interest to compute the order. We do not know any general methods; we can however make the computation in a special case. Namely, let Z_4 act in a free unitary fashion on S^{2n-1}. Now $Z_2 \subset Z_4$, and we obtain a free weakly complex action of Z_2 on $S^{2n-1}/Z_2 = P_{2n-1}(R)$. *We show that the order of $[Z_2, P_{4n+1}(R)]$ is 2^{n+1}; hence by* (6.1) *there exists no equivariant map of $P_{4n+1}(R)$ into S^{2n}*. This represents some progress on a question we have investigated previously [3]. Anderson at Berkeley has proved similar nonexistence theorems by K-theory. Our proof uses a natural homomorphism $\mathfrak{U}_m(B_{S^1}) \to \mathfrak{U}_{m+1}(B_{Z_2})$, and a study of $f_*: \mathfrak{U}_*(B_{S^1}) \to \mathfrak{U}_*(B_{S^1})$ where f acts on a generator $a \in H^2(B_{S^1})$ by $f^*(a) = 2a$.

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UNIVERSITY OF VIRGINIA AND
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