ON THE LOCAL BEHAVIOR OF THE RATIONAL TSCHEBYSCHEFF OPERATOR

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Let l and r be non-negative integers. Denote by $\mathcal{R}_{l,r}$ the set of all rational functions where the degrees of the numerator and denominator do not exceed l and r respectively. If $R=p/q\in\mathcal{R}_{l,r}$ and p and q are relatively prime polynomials of degree ∂p and ∂q, then $d_{l,r}[R]:=\min\,[l-\partial p,\ r-\partial q]$ is called the defect of R in $\mathcal{R}_{l,r}$: the function R is called degenerate, if the defect is positive. (For these notations compare Werner (1962) [**3**].)

For a fixed interval $[a, b]$ let $T_{l,r}[f]$ be the Tschebyscheff Approximation of $f\in C[a, b]$ in the class $\mathcal{R}_{l,r}$ with respect to the norm $\|f\|:=\max_{[a,b]}|w(x)\cdot f(x)|$, with $w(x)$ a positive continuous weight function in $[a, b]$. We write $\eta_{l,r}[f]:=\|f-T_{l,r}[f]\|$. Those f for which $T_{l,r}[f]$ is not degenerate are called normal by Cheney and Loeb (1963) [**1**]. Already Maehly and Witzgall (1960) [**2**] proved that $T_{l,r}[f]$ furnishes a continuous map of $C[a, b]$ into itself at f with respect to the introduced norm, if f is normal. For the actual verification of normality one may use the following *normality criterion*:

Let $g(x)$ be normal for $T_{l,r}$. Then $f(x)$ is normal if

$$\|f-g\| < (\eta_{l-1,r-1}[g] - \eta_{l,r}[g])/2.$$

Except for the case $r=1$, l arbitrary (compare Werner (1963) [**5**]) no specific properties of f are known to insure normality of f for arbitrary l, r.[1] Maehly and Witzgall (1960) [**2**] also gave an example that showed that $T_{l,r}[f]$ need not be continuous at f, if f is not normal. Recently Cheney and Loeb (1963) [**1**] made an extensive study of generalized rational approximation and proved that $T_{l,r}[f]$ is not continuous, if f is not normal and if no alternant of the error function $\eta(x):=w(x)(f(x)-T_{l,r}[f](x))$ has $r+l+2$ points. This later restriction may be lifted and one obtains the following classification.

THEOREM 1. *The operator $T_{l,r}[f]$ is continuous at f if and only if f is normal or belongs to the class $\mathcal{R}_{l,r}$.*

In order to prove this, one now only has to cope with the case that the error function has an alternant of $l+r+2$ points. By a proper

[1] *Added in proof.* Recently a criterion has been published by H. L. Loeb, Notices Amer. Math. Soc. **11** (1964), 335.

construction one finds a sequence of continuous functions f_n; $n=1, 2, \cdots$ that converges uniformly to f and whose associated T-approximations do not converge to $T_{l,r}[f]$.

The construction is not quite easy, because on the other hand one can prove that $T_{l,r}[f_n](x)$ converges to $T_{l,r}[f](x)$ pointwise in (a, b), if f_n converges to f uniformly in $[a, b]$, and if $d_{l,r}[T_{l,r}[f]] \leqq 1$. This result shows that one might expect convergence in a somewhat looser sense. If the defect is greater than 1, then pointwise convergence no longer persists, although from every sequence f_n uniformly converging to f a subsequence can be extracted for which the associated T-approximations converge pointwise with at most r exceptional points in $[a, b]$. Thus the best one can hope for is convergence in measure.

THEOREM 2. *Given $f \in C[a, b]$. To every $\epsilon > 0$, $\epsilon_1 > 0$ one can find $\delta > 0$ such that*

$$\|f - g\| < \delta$$

implies that there is a finite number of intervals depending on g whose total length is less than ϵ_1 such that for all points of $[a, b]$ not lying in the said intervals the inequality

$$| T_{l,r}[f](x) - T_{l,r}[g](x) | < \epsilon$$

holds.

The proofs of these results will be given elsewhere, the methods used are similar to that of §7 of Werner (1962) [4].

REFERENCES

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2. H. J. Maehly and Ch. Witzgall, *Tschebyscheff-Approximationen in kleinen Intervallen*. II, Numer. Math. **2** (1960), 293–307.

3. H. Werner, *Tschebyscheff-Approximation im Bereich der rationalen Funktionen bei Vorliegen einer guten Ausgangsnäherung*, Arch. Rational Mech. Anal. **10** (1962), 205–219.

4. ———, *Die konstruktive Ermittlung der Tschebyscheff-Approximierenden im Bereich der rationalen Funktionen*, Arch. Rational Mech. Anal. **11** (1962), 368–384.

5. ———, *Rationale Tschebyscheff-Approximation, Eigenwerttheorie und Differenzenrechnung*, Arch. Rational Mech. Anal. **13** (1963), 330–347.

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