A PROOF OF THE CORONA CONJECTURE FOR FINITE OPEN RIEMANN SURFACES[1]

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For an open Riemann surface X the corona conjecture is the following: let $B(X)$ be the algebra of bounded analytic functions on X and let $\mathfrak{M}(X)$ be the space of maximal ideals of $B(X)$; then X is dense in $\mathfrak{M}(X)$. Carleson [3] has proved that the corona conjecture is true for the open unit disk D. We will sketch a proof of the following extension of Carleson's Theorem.

THEOREM. *If X is a finite open Riemann surface, then X is dense in $\mathfrak{M}(X)$.*

By a finite open Riemann surface is meant a proper, open, connected subset of a compact Riemann surface W whose boundary Γ is also the boundary of $W-X$ and consists of a finite number of closed analytic arcs. Since $W-X$ has an interior we may employ the Riemann-Roch Theorem to show that $B(X)$ has enough functions to separate points and provide each point in X with a local uniformizer. Such a surface X therefore admits a natural homeomorphic imbedding into $\mathfrak{M}(X)$; thus the corona conjecture is seen to be meaningful.

Let X be a finite open Riemann surface. Ahlfors [1] has shown that there exists an analytic mapping p_0 of $\overline{X}$ into the plane such that $p=p_0|\overline{X}$ is an n-fold covering of X onto D and $p_0(\Gamma)=\overline{D}-D$. Since Γ consists of closed analytic arcs, no ramification occurs on $\overline{D}-D$. Clearly p^*, the adjoint of p, is a C-isomorphism of $B(D)$ into $B(X)$, C being the complex field. Let $B(D)^*$ denote the range of p^*, and for $f\in B(D)$ let $p^*(f)=f^*$.

Let σ_k denote the kth elementary symmetric function on n letters. For $z\in D$ let $p^{-1}(z)=\{x_1(z), \cdots, x_n(z)\}$, each appearing to its multiplicity. Given $f\in B(X)$, $\sigma_k(f(x_1(z)), \cdots, f(x_n(z)))$ is in $B(D)$. Thus, as is well known, $B(X)$ is integrally dependent on $B(D)^*$.

Given $N\in\mathfrak{M}(X)$ let $M^*=N\cap B(D)^*$ and let $P(N)=(p^*)^{-1}(M^*)$. Since $\mathfrak{M}(X)$ and $\mathfrak{M}(D)$ have the weak topology, P is continuous. Further, P is an extension of p. Since $B(X)$ is integrally dependent on $B(D)^*$, P is surjective. For $f\in B(D)((B(X))$ let $\hat{f}$ denote the natural extension of f to $\mathfrak{M}(D)((\mathfrak{M}(X))$. (See Hoffman [4, Chapter 10] for details.) Given $f\in B(D)$, $\hat{f}P=f^{*\frown}$. Let z denote the identity function

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on D and let $\pi = \hat{z}$. Clearly π is a continuous mapping of $\mathfrak{M}(D)$ onto $\overline{D}$, which is the identity on D. We need an analogous function on $\mathfrak{M}(X)$.

Let $B_C(X) = \{f \in B(X) : f$ has a continuous extension on $\overline{X}\}$. Arens [2] has shown that the maximal ideal space of $B_C(X)$ is naturally homeomorphic to $\overline{X}$, with which we will identify it. For $N \in \mathfrak{M}(X)$ let $\Pi(N) = N \cap B_C(X)$. Π is a continuous mapping of $\mathfrak{M}(X)$ onto $\overline{X}$, which is the identity on X. One can show that $p_0 \Pi = \pi P$; thus given $N \in \mathfrak{M}(X)$ such that $P(N) \in D$, N is in X. Let $\gamma \in \Gamma$, $\alpha = p_0(\gamma)$, $\mathfrak{M}(X)_\gamma = \Pi^{-1}(\gamma)$, and let $\mathfrak{M}(D)_\alpha = \pi^{-1}(\alpha)$. Clearly $P(\mathfrak{M}(X)_\gamma) \subset \mathfrak{M}(D)_\alpha$.

Let $K(X)$, $K(D)^*$, $K_C(X)$, and $K_C(D)^*$ be the quotient fields of $B(X)$, $B(D)^*$, $B_C(X)$, and $B_C(D)^*$, respectively, in the field of all meromorphic functions on X. The elementary symmetric functions can be used, as they were above, to define mappings from $K(X)$ into $K(D)^*$ and from $K_C(X)$ into $K_C(D)^*$, with the aid of which it can be shown that $K(X)/K(D)^*$ and $K_C(X)/K_C(D)^*$ are algebraic extensions of degree n. The coefficients of the field polynomial of the extension and the respective symmetric functions are the same. Given $g \in B_C(X)$ and $\alpha \in \overline{D} - D$ such that the continuous extension $\bar{g}$ of g to $\overline{X}$ assumes n distinct values on $\bar{p}_0^{-1}(\alpha)$, then g generates $K(X)$, $K_C(X)$ over $K(D)^*$, $K_C(D)^*$ respectively. The discriminate d^* of g is in $B_C(D)^*$. Its extension $\bar{d}$ to $\overline{D}$ is nonzero at α, since g assumes distinct values on $\bar{p}_0^{-1}(\alpha)$. Since g generates $K(X)$ over $K(D)^*$ and $K_C(X)$ over $K_C(D)^*$, given $b \in B(X)$ $((B_C(X))$ there exist unique $f_0, \cdots, f_{n-1} \in K(D)$ $(K_C(D))$ such that $b = \sum_{j=0}^{n-1} f_j^* g^j$. Using Cramer's rule, one can find unique $a_0, \cdots, a_{n-1} \in B(D)$ $(B_C(D))$ such that $f_j = a_j/d$. With the aid of this elementary field theory the following can be proved: $P | \mathfrak{M}(X)_\gamma$ is an injection onto $\mathfrak{M}(D)_\alpha$.

By the nature of p_0, we can choose a closed neighborhood V of α in $\overline{D}$ such that if U is the component of $\bar{p}_0^{-1}(V) \cap \overline{X}$ that contains γ, then $p_0 | U$ is a homeomorphism. Let $\mathcal{V} = \pi^{-1}(V)$ and let $\mathcal{U} = \Pi^{-1}(U)$. Since $\mathfrak{M}(X)$ and $\mathfrak{M}(D)$ are compact, $\mathcal{U}$ and $\mathcal{V}$ are compact. Since $P | \mathfrak{M}(X)_\gamma$ is an injective mapping onto $\mathfrak{M}(D)_\alpha$, and by the choice of U and V, $P | \mathcal{U}$ is an injection onto $\mathcal{V}$. Since $\mathcal{U}$ is compact $P | \mathcal{U}$ is a homeomorphism. From this it follows easily that P is an open mapping. Invoking Carleson's Theorem [3], for the first time, we find that X is dense in $\mathfrak{M}(X)$, proving the theorem.

Actually, we have proved somewhat more. Let $\Gamma_1, \cdots, \Gamma_k$ be the components of Γ. By definition, these components are nondegenerate, closed, analytic curves in W that p_0 takes to S, the unit circle. Since $\mathfrak{M}(D)_\alpha$ is connected for each $\alpha \in S$ [4], $\mathfrak{M}(X)_\gamma$ is connected for each $\gamma \in \Gamma$; thus $\mathfrak{M}(X)_{\Gamma_i} (= \Pi^{-1}(\Gamma_i))$ is connected for each i. We conclude that $\mathfrak{M}(X)_{\Gamma_1}, \cdots, \mathfrak{M}(X)_{\Gamma_k}$ are the components of $\mathfrak{M}(X)-X$. Let q_i be the number of times Γ_i covers S under p_0; clearly $\sum_{i=1}^{k} q_i$ equals n. $P|\mathfrak{M}(X)_{\Gamma_i}$ is a local homeomorphism of $\mathfrak{M}(X)_{\Gamma_i}$ onto $\mathfrak{M}(D)-D$, that covers each point q_i times.

Let Y be an open Riemann surface and suppose there exists a conformal homeomorphism of Y onto a finite open Riemann surface X. Then clearly we can carry the solution of the corona conjecture to Y.

REMARK. Using recent results of Röhrl [5, Theorem 4.2], one can show that $B(X)$ is a free $B(D)^*$-module of dimension n, a basis for which can be chosen in $B_C(X)$. With this one can show that $P|\mathfrak{M}(X)_\gamma$ is injective. It seems, at this time, not unreasonable to conjecture that an element $g \in B_C(X)$ can be found such that $1, g, \cdots, g^{n-1}$ is a free basis of $B(X)$ over $B(D)^*$. From this it would follow immediately that $P(\mathfrak{M}(X)_\gamma)=\mathfrak{M}(D)_\alpha$ (this can also be shown using Carleson's Theorem).

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