ON THE LOCALIZATION AND DIRECTIONALIZATION OF UNIFORM CONVEXITY[1]

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1. Introduction. There are two main theorems in this note. The first states the close connection between the localization and directionalization of uniform convexity in a normed space X and various differentiability conditions satisfied by the norm in X^*. This answers a question implicit in [1; 5] and makes possible the extension of the dual theory of differentiability of the norm as initiated in [6; 7]. The second main theorem examines the spherical image map defined on certain infinite dimensional manifolds imbedded in X and gives characterizations of the smoothness of the manifold in terms of the continuity properties of the spherical image map in various topologies in the spaces X and X^*. These results can be used to solve a problem proposed in [4].

Notation. X will be an arbitrary normed linear space with the reals as scalar field. $U=\{x: \|x\| \leqq 1,\ x \in X\}$, $S=\{x: \|x\|=1,\ x\in X\}$ and U' and S' denote the analogous sets in X^*. Q is the canonical map imbedding X into X^{**}. When it is to be emphasized that the conjugate norm in X^* is being considered, this norm will be denoted by $\|\cdot\|^*$. θ is the neutral element of a linear space.

2. Modifications of uniform convexity and differentiability of the norm. Define the modulus of uniform convexity by

$$\delta(\epsilon) = \inf_{\|x-y\| \geqq \epsilon;\, x,y\in S} 2 - \|x+y\|.$$

In the next few definitions let f and g be fixed elements in S'. Then the modulus of *weak uniform convexity at* f is defined by

$$\delta(\epsilon, f) = \inf_{\|x-y\| \geqq \epsilon;\, x,y\in S} 2 - |f(x+y)|;$$

the modulus of *uniform convexity in the direction* g is defined by

$$\delta(\epsilon, g) = \inf_{|g(x-y)| \geqq \epsilon;\, x,y\in S} 2 - \|x+y\|;$$

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and the modulus of *weak uniform convexity at f in the direction g* is defined by

$$\delta(\epsilon, f, g) = \inf_{|g(x-y)| \geqq \epsilon; x, y \in S} 2 - |f(x+y)|.$$

X will be said to be *uniformly convex* (see [2]), *weakly uniformly convex at f*, *uniformly convex in the direction g*, or *weakly uniformly convex at f in the direction g* provided the corresponding modulus is positive when $0 < \epsilon < 2$.

Let $x, y \in S$ and consider the limit

$$(*) \qquad G^+(x, y) = \lim_{\alpha \to 0^+} \frac{\|x + \alpha y\| - \|x\|}{\alpha}.$$

Let $G^-(x, y) = -G^+(x, -y)$. Then $\|\cdot\|$ is said to be *differentiable at x with respect to y* provided $G^+(x, y) = G^-(x, y)$; $\|\cdot\|$ is said to be *Fréchet differentiable at x* provided $G^+(x, y) = G^-(x, y)$ for all y in S and the limit (*) is approached uniformly as y varies over S; $\|\cdot\|$ is said to be *uniformly Gateaux differentiable in the direction y* provided $G^+(x, y) = G^-(x, y)$ for all x in S and the limit (*) is approached uniformly as x varies over S; $\|\cdot\|$ is said to be *uniformly Fréchet differentiable* provided $G^+(x, y) = G^-(x, y)$ for all x and y in S and the limit (*) is approached uniformly as x and y vary over S.

In the next theorem X is an arbitrary normed linear space over the reals and $\|x\| = \|y\| = \|f\|^* = \|g\|^* = 1$. Dual results are indicated by primes. Šmulian [7] gives (iv) and (iv′) and the sequentialized forms of (ii) and (ii′).

THEOREM 1. (i) $\|\cdot\|^*$ *is differentiable at f with respect to g iff X is weakly uniformly convex at f in the direction g.* (ii) $\|\cdot\|^*$ *is Fréchet differentiable at f iff X is weakly uniformly convex at f.* (iii) $\|\cdot\|^*$ *is uniformly Gateaux differentiable in the direction g iff X is uniformly convex in the direction g.* (iv) $\|\cdot\|^*$ *is uniformly Fréchet differentiable iff X is uniformly convex.* (i′) $\|\cdot\|$ *is differentiable at x with respect to y iff X^* is weakly uniformly convex at Qx in the direction Qy.* (ii′) $\|\cdot\|$ *is Fréchet differentiable at x iff X^* is weakly uniformly convex at Qx.* (iii′) $\|\cdot\|$ *is uniformly Gateaux differentiable in the direction y iff X^* is uniformly convex in the direction Qy.* (iv′) $\|\cdot\|$ *is uniformly Fréchet differentiable iff X^* is uniformly convex.*

3. The spherical image map. Let K be a closed bounded convex set with an interior point in the norm topology of X. For each point x on the boundary of K let ν_x, the *spherical image of x*, be the set of linear functionals of norm one with the property that the inverse image of a positive number is a hyperplane of support of K at x.

Denote by $\|\cdot\|_1$ a norm equivalent to $\|\cdot\|$ and let $U_1=\{x: \|x\|_1 \leqq 1, x \in X\}$, $S_1=\{x: \|x\|_1=1, x\in X\}$. In the next theorem the infinite dimensional manifolds S and S_1 are considered and in the latter case the spherical image map ν is defined as above by setting $U_1=K$. The "only if" part of (i) is given in [6].

THEOREM 2. (i) *$\|\cdot\|$ is differentiable at any x on S with respect to any y on S iff the map ν defined on S is lower semicontinuous from the norm topology on S into the weak* topology on S'.* (ii) *$\|\cdot\|$ is uniformly Gateaux differentiable in any direction y on S iff the map ν defined on S is single valued and uniformly continuous from the norm topology on S into the weak* topology on S'.* (iii) *Let ν be defined on S_1. Then $\|\cdot\|_1$ is [uniformly] Fréchet differentiable iff ν is single valued and [uniformly] continuous from the norm topology on S_1 into the norm topology on S'.*

4. **Application.** Define the *extended spherical image map* T from X into X^* by setting $T\theta=\theta$ and $Tx=\|x\|\nu_{\|x\|^{-1}x}$ for $x\neq\theta$. It is well known that if X is a real Hilbert space (so that X is uniformly convex and uniformly Fréchet differentiable) then T is a linear isometry between X and X^*. Klee [4, p. 35] has posed the following problem: Characterize intrinsically those spaces X for which T is a homeomorphism of X onto X^* in the norm topologies.

The main theorems of this note can be combined to give the following answer to this problem.

THEOREM 3. *T is a homeomorphism iff X is a weakly uniformly convex Banach space with a Fréchet differentiable norm.*

The proofs of these and other theorems with other applications are to be found in [3].

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