CONJECTURE. Every special vector lattice of self-adjoint elements in the regular ring of a finite AW*-algebra is commutative.

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REMARK ON MY PAPER "SIMULTANEOUS APPROXIMATION AND ALGEBRAIC INDEPENDENCE OF NUMBERS"

BY W. M. SCHMIDT

It has been pointed out to me that a result very similar to the one proved in my paper was obtained by O. Perron, *Über mehrfach transzendente Erweiterungen des natürlichen Rationalitätsbereiches*, Sitzungsberichte Bayer. Akad. Wiss. H2 (1932), 79–86.

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INTEGRAL NORMS OF SUBADDITIVE FUNCTIONS

BY R. P. GOSSELIN[1]

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It is known that certain integral norms for positive, measurable, subadditive functions of a single variable are comparable (cf. [1]; also [3; 4] for less complete results). This fact was shown to have

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applications for Lipschitz classes and for fractional integrals. Analogous results are true in higher dimensions as we show here. Moreover, a new phenomenon appears: an integral norm taken over E^n, n-dimensional Euclidean space, is shown to be equivalent to one taken over certain one dimensional spaces. It is by means of this last theorem, which we take to be the main result of the paper, that we obtain inequalities analogous to those for one dimension. Further applications, related to results of [2; 5], as well as variants of the results given below, will be discussed elsewhere.

In the matter of notation, x will stand for a point of E^n, r a real, Ω the unit sphere, and ω a unit vector, i.e. a point of Ω. For a measurable set G of E^n, $|G|$ will denote its Lebesgue measure and, for a measurable set Γ of Ω, $|\Gamma|_s$ will denote its spherical measure. Subadditivity of a function ϕ over E^n means that for x and y in E^n, $\phi(x+y) \leqq \phi(x)+\phi(y)$.

THEOREM. *Let ϕ be positive, measurable, and subadditive on E^n, $n>1$. Let $\omega_1, \omega_2, \cdots, \omega_n$ be linearly independent unit vectors, and let $0<p<\infty$ and $\alpha p>-1$. There exist constants A and B depending only on α, p, and n such that*

$$\int_{E^n} \frac{\phi^p(x)}{|x|^{n+p\alpha}} dx \leqq A \sum_{i=1}^{n} \int_{E^1} \frac{\phi^p(r\omega_i)}{|r|^{1+p\alpha}} dr \leqq B \int_{E^n} \frac{\phi^p(x)}{|x|^{n+p\alpha}} dx.$$

We may omit the restriction that $\alpha p>-1$ for the second inequality. A special case of the above theorem is known for ϕ a certain integral norm [5]. The proof of the second inequality relies heavily on the following.

LEMMA. *There exist constants a, b, and c with $0<a<b<\infty$ and $0<c<1$ such that if Γ is a measurable subset of Ω for which $|\Gamma|_s \geqq c|\Omega|_s$, then for every ω of Ω there exist r_1 and r_2 with $a \leqq r_1, r_2 \leqq b$ and ω_1, ω_2 both in Γ such that $\omega = r_1\omega_1 + r_2\omega_2$.*

Fix α and β such that $0<\alpha<\beta<1$, and let c satisfy

$$\frac{\beta^n - \alpha^n}{(1+\beta)^n} > \frac{1-c}{c}, \qquad 0 < c < 1. \tag{1}$$

Let the complement (in Ω) of Γ be denoted by Γ'. Then $|\Gamma'|_s \leqq (1-c)|\Omega|_s$ and so $|\Gamma|_s \geqq |\Gamma'|_s c/(1-c)$. Now let G denote the set of x of the form $r\omega$ where ω belongs to Γ and $\alpha \leqq r \leqq \beta$. Fix ω_0 on Ω and let $H=\omega_0-G$, i.e. the set of points of the form ω_0-x with x in G. Then

$$|H| = |G| = |\Gamma|_s \frac{\beta^n - \alpha^n}{n}. \tag{2}$$

Let Λ be the radial projection of H onto Ω, and let Λ have outer spherical measure h. Let Λ_1 be a measurable subset of Ω containing Λ and such that $|\Lambda_1|_s = h$. Let $\chi(r\omega)$, $\psi(\omega)$, and $\psi_1(\omega)$ be the characteristic functions of H, Λ, and Λ_1 respectively. Now $\psi(\omega)=1$ if and only if $\chi(r\omega)=1$ for some $r\neq 0$. Thus

$$\chi(r\omega) \leqq \psi(\omega) \leqq \psi_1(\omega), \qquad r \neq 0.$$

Since $\chi(r\omega)=0$ if $r>1+\beta$, integration of the extreme terms of the preceding inequality over the solid sphere of radius $1+\beta$ gives

$$|H| \leqq \frac{(1+\beta)^n}{n} |\Lambda_1|_s.$$

(1) and (2) imply

$$|\Gamma'|_s \leqq \frac{(1+\beta)^n}{\beta^n-\alpha^n}\,\frac{1-c}{c}\,|\Lambda_1|_s < |\Lambda_1|_s$$

since obviously $|\Lambda_1|_s\neq 0$. This implies that Λ is not a subset of Γ', and so there exists ω_1 in $\Gamma\cap\Lambda$. Hence $r_1\omega_1$ belongs to H for some $r_1>0$, and $\omega_0-r_1\omega_1=r_2\omega_2$ belongs to G. Thus ω_2 also belongs to Γ, and since $\alpha\leqq r_2\leqq\beta$, we have $1-\beta\leqq r_1\leqq 1+\beta$. Thus, min $(\alpha, 1-\beta)\leqq r_1, r_2\leqq 1+\beta$, and the extreme terms of this inequality can be taken for a and b respectively.

For the second inequality of the theorem, it is enough to show the existence of a constant C such that for every ω of Ω

$$M(\omega) \leqq CM \tag{3}$$

where

$$M = \int_{E^n} \frac{\phi^p(x)}{|x|^{n+p\alpha}}dx = \int_\Omega d\omega \int_0^\infty \frac{\phi^p(r\omega)}{r^{1+p\alpha}}dr = \int_\Omega M(\omega)d\omega.$$

We may assume that M is finite and strictly positive. Let Γ be the set of ω such that $M(\omega)\leqq M/(1-c)|\Omega|_s$ where c is defined in the lemma. Γ is a measurable set of Ω, and $|\Gamma|_s\geqq c|\Omega|_s$. Thus the lemma applies and we may assert that for each ω of Ω, there exist r_i, $a\leqq r_i\leqq b$, and ω_i in Γ, $i=1, 2$, such that $\omega=r_1\omega_1+r_2\omega_2$. It follows from the subadditivity property and from Minkowski's inequality (modified if $p<1$) that

$$\phi^p(r\omega) \leqq C\{\phi^p(rr_1\omega_1) + \phi^p(rr_2\omega_2)\}.$$

Integration gives

$$M(\omega) \leqq C\{r_1^{\alpha p}M(\omega_1) + r_2^{\alpha p}M(\omega_2)\}.$$

If $\alpha \geqq 0$, $r_i^{\alpha p} \leqq b^{\alpha p}$, $i=1, 2$; if $\alpha<0$, $r_i^{\alpha p} \leqq a^{\alpha p}$, $i=1, 2$. Since $M(\omega_i) \leqq CM$, the proof of the second inequality is complete.

Now given $x \neq 0$, let $x = r\omega$. We write $\omega = \sum_{i=1}^n c_i\omega_i$ where the ω_i are as given by the theorem and $c_i = c_i(\omega)$. By subadditivity and Minkowski's inequality

$$\phi^p(r\omega) \leqq C \sum_{i=1}^n \phi^p(c_i r\omega_i).$$

Dividing by $r^{1+p\alpha}$ and integrating, we obtain

$$\int_0^\infty \frac{\phi^p(r\omega)}{r^{1+p\alpha}} dr \leqq C \sum_{i=1}^n \left| c_i(\omega) \right|^{p\alpha} \int_{-\infty}^\infty \frac{\phi^p(r\omega_i)}{\left| r \right|^{1+p\alpha}} dr.$$

An integration with respect to ω over Ω then shows that if the finiteness of the integrals $\int_\Omega |c_i(\omega)|^p d\omega$, $i=1, 2, \cdots, n$, is proved, then the theorem is proved. For this last step, we fix i. By a rotation, ω_i may be made to coincide with the positive direction of the first coordinate axis of E^n. Then $c_i(\omega)$ equals the first coordinate of ω, and in terms of spherical coordinates, this is $\cos\psi$, say. The integrability of $|c_i(\omega)|^{p\alpha}$ over Ω is then equivalent to that of $|\cos\psi|^{p\alpha}\sin^{n-2}\psi$ over $(0, \pi)$.

COROLLARY. *Let ϕ be positive, measurable, and subadditive on E^n. Let α be real, and $1 \leqq p < q \leqq \infty$. There exists a constant $A = A(\alpha, p, q, n)$ such that*

$$\left(\int_{E^n} \frac{\phi^q(x)}{\left| x \right|^{n+q\alpha}} dx\right)^{1/q} \leqq A\left(\int_{E^n} \frac{\phi^p(x)}{\left| x \right|^{n+p\alpha}} dx\right)^{1/p}.$$

Let $M^{1/p}$ denote the integral on the right. It may be assumed finite and strictly positive. Using the notation of (3), we have $M(\omega) \leqq CM$ for every ω of Ω. Assume $q < \infty$, and let

$$M_1(\omega) = \int_0^\infty \frac{\phi^q(r\omega)}{r^{1+q\alpha}} dr.$$

Since $\phi(r\omega)$ is subadditive in r over $(0, \infty)$, by the one dimensional result, $M_1^{1/q}(\omega) \leqq CM^{1/p}(\omega)$. (Cf. [1] where the result was stated only for $p=1$. The proof remains valid in this case.) Thus by (3), $M_1(\omega) \leqq CM^{q/p-1}M(\omega)$. Integration over Ω of this inequality completes the proof for the case $q < \infty$. For $q = \infty$, the left integral of the corollary is to be interpreted as $\sup \phi(x)/|x|^\alpha$. Again the one dimensional result along with (3) suffice for the proof.

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ON RAISING FLOWS AND MAPPINGS

BY R. D. ANDERSON[1]

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It is assumed that all spaces with which we are concerned are separable metric. Let (G, X) be a transformation group with $G=I, R$ where I is the additive group of integers and R the reals. (I, X) is called a *discrete flow* and (R, X) a *continuous flow*. The *orbit* O_x of a point $x \in X$ under a flow (G, X) is the set of all gx for $g \in G$. A flow (G, Y) is *imbedded* in a flow (G, Y') if $Y \subset Y'$ and (G, Y) is (G, Y') cut down to Y. We say that (G, X) is *raised* to (G, Y) provided there is a mapping ϕ of Y onto X such that for each $y \in Y$ and $g \in G$, $\phi g(y) = g\phi(y)$. In this paper we establish that any discrete flow can be raised to a discrete flow on a zero-dimensional space and any continuous flow to a continuous flow on a 1-dimensional space. We shall note that these newly produced flows can be considered as imbedded in a discrete flow on the disc on the one hand and in a continuous flow on the solid torus in Euclidean 3-space on the other. Thus all continuous flows on compact metric spaces can be produced from continuous flows on the solid torus. We include some remarks about minimal flows in §3.

1. **Discrete flows.** Any homeomorphism g of X onto X generates a discrete flow on X and in turn any discrete flow is generated by the unit of the group G. Let g_x denote the unit of the group or the generating homeomorphism.

We wish to establish

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