ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

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1. Classification of certain spaces of continuously differentiable functions of two variables. Denote by C_0 the space of all complex-valued continuous functions on the plane that are zero at infinity. Write $\|\cdot\|_\infty$ for the supremum norm on C_0. Denote by D the dense subspace of C_0 consisting of infinitely differentiable functions with compact support.

Throughout we shall be concerned with differential operators of the form

$$\sum a_{m,n} \frac{\partial^{m+n}}{\partial x^m \partial y^n}; \tag{1.1}$$

the $a_{m,n}$ are complex constants. For each set $\mathfrak{A}$ of such operators, we define $C_0(\mathfrak{A})$ to be the space of all f in C_0 having Af in C_0 (in the sense of Laurent Schwartz) for all A in $\mathfrak{A}$. Equivalently, $C_0(\mathfrak{A})$ is the completion of D under the seminorms

$$f \to \|f\|_\infty \quad \text{and} \quad f \to \|Af\|_\infty, \qquad A \text{ in } \mathfrak{A}.$$

Each $C_0(\mathfrak{A})$ so defined is a translation-invariant space of functions; those that are furthermore invariant under rotations of the plane will be called *rotating spaces of differentiable functions.*

Certain of these spaces are familiar, namely the spaces C_0^N consisting of those functions in C_0 that have all derivatives of order $\leqq N$ in C_0, and the space C_0^∞, which is $\bigcap_N C_0^N$. A rotating space of differentiable functions will be called *proper* if it is not one of the C_0^N and not C_0^∞. Here is the classification of such spaces.

We use the notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

THEOREM 1.1. *If $\mathfrak{A}$ is a proper subset of*

$$\left\{\partial^{m+n}/\partial z^m \partial \bar{z}^n : m + n = N\right\}, \tag{1.2}$$

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for N a positive integer, then $C_0(\mathfrak{A})$ is a proper rotating space of differentiable functions between C_0^N and C_0^{N-1}. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are distinct subsets of (1.2) *then $C_0(\mathfrak{A}_1)$ and $C_0(\mathfrak{A}_2)$ are distinct. Each proper rotating space of differentiable functions is a $C_0(\mathfrak{A})$, where $\mathfrak{A}$ is a proper subset of* (1.2) *for some N.*

COROLLARY 1.2. *Every proper rotating space of differentiable functions is a Banach algebra under pointwise multiplication.*

For instance, the six distinct proper rotating spaces of differentiable functions between C_0^1 and C_0^2 are

$$\begin{aligned}
&C_0(\partial^2/\partial z^2),\\
&C_0(\partial^2/\partial \bar{z}^2),\\
&C_0(\partial^2/\partial z\partial\bar{z}) = C_0(\partial^2/\partial x^2 + \partial^2/\partial y^2),\\
&C_0(\partial^2/\partial z^2,\ \partial^2/\partial \bar{z}^2) = C_0(\partial^2/\partial x^2 - \partial^2/\partial y^2,\ \partial^2/\partial x\partial y),\\
&C_0(\partial^2/\partial z^2,\ \partial^2/\partial z\partial\bar{z}),\\
&C_0(\partial^2/\partial \bar{z}^2,\ \partial^2/\partial z\partial\bar{z}).
\end{aligned}$$

REMARK 1.3. *Suppose instead of the supremum norm we take as basic norm $\|f\|_p = \{\int |f|^p\}^{1/p}$, $1<p<\infty$. Then there are no proper rotating spaces*

$$L_p(\mathfrak{A}) = \{f : f \in L_p,\ Af \in L_p \text{ for all } A \in \mathfrak{A}\}.$$

Indeed the only rotating spaces are the Sobolev spaces

$$L_p^N = \{f : f \in L_p,\ (\partial^{m+n}/\partial x^m \partial y^n) f \in L_p,\ m+n \leqq N\},$$

analogs of the C_0^N; and L_p^∞, which is identical with C_0^∞.

In a sense, rotating spaces are the ones that have geometrical significance. From this point of view the correct definition in norm $\|\cdot\|_\infty$ of "Sobolev space" would include not only the C_0^N but also the proper rotating spaces.

2. **Spaces of continuously differentiable functions on Riemann surfaces.** It is possible to define algebras of functions on Riemann surfaces corresponding to those described in Theorem 1.1 and to show that these algebras determine the conformal structure.

Let U be an open subset of the plane, $C(U)$ the space of all complex-valued continuous functions on U. For $\mathfrak{A}$ a set of differential operators of the form (1.1), we denote by $C(U, \mathfrak{A})$ the subspace of $C(U)$ consisting of those f in $C(U)$ with Af in $C(U)$ (in the sense of Laurent Schwartz) for each A in $\mathfrak{A}$.

For a general set $\mathfrak{A}$ of differential operators, the spaces $C(U, \mathfrak{A})$

have no interesting invariance properties. However we have

LEMMA 2.1. *If* $\mathfrak{A}$ *is a subset of* (1.2), *the spaces* $C(U, \mathfrak{A})$ *are invariant under conformal transformations.*

This result allows the extension of the definition of the $C(U, \mathfrak{A})$ to Riemann surfaces. If R is a Riemann surface and $\mathfrak{A}$ a subset of (1.2), $C(R, \mathfrak{A})$ is defined to be the space of those functions on R such that if $U = \{z: |z| < 1\}$ and $\phi: U \to R$ is a coordinate disk, the composite function $f \circ \phi$ is in $C(U, \mathfrak{A})$.

Each $C(R, \mathfrak{A})$ is an algebra of functions on R, with multiplicative linear functionals corresponding to points of R, and with a natural complete locally convex topology. In this topology $C(R, \mathfrak{A})$ is a Banach algebra if and only if R is compact.

In three instances $C(R, \mathfrak{A})$ can be described in terms of exterior differential operators defined globally on R.

$$C(R, \partial/\partial z) = \{f: f \text{ and } \partial f \text{ continuous on } R\},$$

$$C(R, \partial/\partial \bar{z}) = \{f: f \text{ and } \bar{\partial} f \text{ continuous on } R\},$$

$$C(R, \partial^2/\partial z \partial \bar{z}) = \{f: f \text{ and } \Delta f \text{ continuous on } R\},$$

where the operators ∂, $\bar{\partial}$ and Δ, taking functions into differential forms, are defined in terms of any coordinate system by

$$\partial f = \frac{\partial f}{\partial z}\, dz, \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}}\, d\bar{z}, \quad \Delta f = \frac{1}{4}\, \frac{\partial^2 f}{\partial z \partial \bar{z}}\, dz d\bar{z}.$$

The following result states the extent to which the algebras $C(R, \mathfrak{A})$ determine the conformal structure of R.

THEOREM 2.2. *Let* R_1 *and* R_2 *be connected Riemann surfaces and* $\mathfrak{A}$ *be a proper subset of* (1.2). *Each conformal equivalence of* R_1 *with* R_2 *induces an algebra isomorphism of* $C(R_1, \mathfrak{A})$ *with* $C(R_2, \mathfrak{A})$. *If* $\mathfrak{A}$ *is symmetric* (*i.e.*, $\partial^{m+n}/\partial z^m \partial \bar{z}^n$ *in* $\mathfrak{A}$ *if* $\partial^{m+n}/\partial z^n \partial \bar{z}^m$ *in* $\mathfrak{A}$), *an anticonformal equivalence of* R_1 *with* R_2 *also induces an algebra isomorphism of* $C(R_1, \mathfrak{A})$ *with* $C(R_2, \mathfrak{A})$. *No other algebra isomorphisms of* $C(R_1, \mathfrak{A})$ *with* $C(R_2, \mathfrak{A})$ *are possible.*

A similar result identifies all algebra homomorphisms of the $C(R, \mathfrak{A})$.

3. **Sup norm estimates.** The work of the preceding sections is based on the existence and nonexistence of certain sup norm estimates for constant-coefficient differential operators. In this section we state

these results, which may be of some independent interest. All these results remain valid for n variables.

If P is a polynomial,

$$P(x, y) = \sum a_{m,n} x^m y^n,$$

we denote by P^N its homogeneous part of degree N,

$$P^N(x, y) = \sum_{m+n=N} a_{m,n} x^m y^n,$$

and by $\hat{P}$ its Fourier transform, the differential operator

$$\sum (-i)^{m+n} a_{m,n} \frac{\partial^{m+n}}{\partial x^m \partial y^n} \cdot$$

An operator $\hat{P}$ of order N is called *elliptic* if $P^N(x, y) \neq 0$ for $(x, y) \neq (0, 0)$.

THEOREM 3.1. *Let $Q, P_1, \cdots, P_r$ be polynomials of degree N or less. Then the following are equivalent*:

(1) *There is a constant K so that*

$$\|\hat{Q}f\|_\infty \leqq K(\|\hat{P}_1 f\|_\infty + \cdots + \|\hat{P}_r f\|_\infty)$$

for all f in D.

(2) *There are finite measures $\mu_1, \cdots, \mu_r$ in the plane whose Fourier-Stieltjes transforms $\hat{\mu}_1, \cdots, \hat{\mu}_r$ satisfy*

$$Q = P_1\hat{\mu}_1 + \cdots + P_r\hat{\mu}_r.$$

If (1) *and* (2) *hold, it is possible to find constants $c_1, \cdots, c_r$ so that*

$$Q^N = c_1 P_1^N + \cdots + c_r P_r^N.$$

THEOREM 3.2. *Let $\hat{P}$ be elliptic and of order $N \geqq 2$. Then for each $\hat{Q}$ of order strictly less than N there is a constant K so that*

$$\|\hat{Q}f\|_\infty \leqq K(\|\hat{P}f\|_\infty + \|f\|_\infty)$$

for all f in D.

COROLLARY 3.3. $\partial^{m+n}/\partial z^m \partial \bar{z}^n$ *is elliptic. Hence, if $p+q<m+n$, there is a constant K so that*

$$\|(\partial^{p+q}/\partial x^p \partial y^q) f\|_\infty \leqq K(\|(\partial^{m+n}/\partial z^m \partial \bar{z}^n) f\|_\infty + \|f\|_\infty)$$

for all f in D.

REMARK 3.4. The situation for estimates in $\|\cdot\|_p$, $1<p<\infty$, is quite different, which is the reason for the phenomenon mentioned in Remark 1.3. To be precise, if $\hat{P}$ is elliptic of order N, $\hat{Q}$ arbitrary of order $\leqq N$, it is known that there is an estimate of the form

$$\|\hat{Q}f\|_p \leqq K(\|\hat{P}f\|_p + \|f\|_p) \qquad f \in D,$$

and the existence of these estimates is characteristic of ellipticity. It is natural to ask whether the existence of sup norm estimates like those in Theorem 3.2 (where $\hat{Q}$ has strictly lower order) also characterizes ellipticity. In the plane the answer is no, but in higher dimensions yes.

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