RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

COMBINATORIAL EMBEDDINGS OF MANIFOLDS[1]

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The following results on embedding manifolds resemble in their form Dehn's Lemma, the Sphere Theorem, and, especially, embedding theorems obtained for differentiable manifolds by A. Haefliger [1].

Let M, Q be finite combinatorial manifolds of dimensions m and q, respectively. Let $\dot{M}$, $\dot{Q}$ be their boundaries (possibly empty), and let $f\colon M \to Q$ be a piecewise linear map. We define *sing* (f) to be the closure in M of the set $\{x \in M; f^{-1}f(x) \neq x\}$. Let $R = \dot{M} \cap f^{-1}(\dot{Q})$, S be a regular neighbourhood of R in $\dot{M}$ (see [3]) and $T = \dot{M} - \mathring{S}$.

THEOREM 1. *Of the following conditions,* (i), (ii), (iii), *and any one of* (iv), (v), (vi) *are sufficient to ensure the existence of a piecewise linear embedding* $g\colon M \subset Q$ *such that* g *is homotopic to* f *rel.* $\dot{M}$:

(i) $q \geqq m+3$,

(ii) M *is* $(2m-q)$ *connected,*

(iii) Q *is* $(2m-q+1)$ *connected,*

(iv) $f(\dot{M}) \subset \dot{Q}$,

(v) $sing(f) \cap \dot{M} = \emptyset$ *and* T *is* $(3m-2q+1)$ *connected,*

(vi) $sing(f) \cap R = \emptyset$ *and* T *is* $(2m-q-1)$ *connected.*

REMARKS. If $\dot{M} = \emptyset$, we regard condition (iv) as being trivially satisfied. If $f(\dot{M}) \subset \mathring{Q}$, we have the convention that the only regular neighbourhood of the empty set is the empty set, and so $T = \dot{M}$.

In particular:

COROLLARY 2. *Any element of* $\Pi_m(Q)$, *where* Q *is* $(2m-q+1)$ *connected* $(q \geqq m+3)$, *may be represented by a piecewise linear embedding of* S^m.

THEOREM 3. *If* $q = 2m$, *there exists a piecewise linear embedding*

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$g: M \subset Q$ such that g is homotopic to f rel. $\dot{M}$ provided that M is connected, $\dot{M}$ is non empty and $f(\dot{M})$ is not contained in $\dot{Q}$.

We recall that if X, X_0 are two complexes, X is said to contract to X_0 *by an elementary contraction* if $X = X_0 + \sigma + \tau$ where σ and τ are simplexes (boundaries $\dot{\sigma}$ and $\dot{\tau}$ respectively), $\sigma = a\tau$ (a a vertex), $a\dot{\tau} \in X_0$, and $\tau \notin X_0$. We say that, if X, X_0 are subcomplexes of a combinatorial manifold A, X contracts to X_0 *by an admissible elementary contraction in* A if X contracts to X_0 by an elementary contraction $X = X_0 + \sigma + \tau$, and the following case does *not* hold: $\tau \in \dot{A}$, $\sigma \notin \dot{A}$ ($\dot{A}$ is the boundary of A). We say that X is *admissibly collapsible* in A if X contracts to a point by a series of admissible elementary contractions in A.

The proof of Theorem 1 leans heavily on the following Lemma, which is an extension of a result of E. C. Zeeman [**5**; **6**] to cover the case of a bounded manifold.

LEMMA. *If A is a t-dimensional finite combinatorial manifold with boundary $\dot{A}$, X an admissibly collapsible subcomplex of A, and K an s-dimensional subcomplex of A such that $K \cap \dot{A} = J$, and if further*

(i) *J is r-dimensional, $r < s$,*

(ii) *$t \geqq s+3$,*

(iii) *A is s-connected and $\dot{A}$ r-connected,*

then, in a suitable subdivision βA of A, there exists an $(s+1)$-dimensional subcomplex L of βA such that $\beta K \subset L$, and $\beta X \cup L$ is admissibly collapsible in βA.

Theorem 3 is proved by an argument practically identical to that of Lemma 2.7 in [**2**].

There are counterexamples to show the necessity for some, perhaps weaker, conditions like (ii) and (iii) of Theorem 1.

COUNTEREXAMPLE 1. (Condition (ii) for M closed.) M = real projective space of dimension 2^r, $r > 0$, Q = a combinatorial q-ball B^q, where $q < 2 \cdot 2^r$, f = the map taking M to a point of Q. It is well known that there is no embedding of M in Q.

COUNTEREXAMPLE 2. (Condition (ii) for M bounded.) $M = S^r \times B^{s+1}$, $\dot{M} = S^r \times S^s$, $Q = B^{2s+2}$, $\dot{Q} = S^{2s+1}$, where r and s are such that S^s admits a continuous r-field of tangent vectors, f is a piecewise linear map of M in Q, taking $\dot{M}$ into $\dot{Q}$ in such a way that two cross-sectional copies of S^s become homologically once linked in S^{2s+1} (see [**4**; **6**]). There exists no embedding g of M in Q with $g | \dot{M} = f | \dot{M}$ for

SUBLEMMA. *If S_1^s, $S_2^s \subset S^{2s+1}$ are homologically once linked combina-*

torial s-spheres, there do not exist in B^{2s+2} *combinatorial* $(s+1)$*-balls* B_1^{s+1}, B_2^{s+1} *spanning* S_1^s, S_2^s *and such that* $B_1^{s+1} \cap B_2^{s+1} = \emptyset$.

COUNTEREXAMPLE 3. (Condition (iii)). There exists a $2m$-dimensional orientable finite combinatorial manifold $Q(m \geqq 2)$ such that $\Pi_m(Q)$ is nonzero, and no nonzero element of it can be represented by a piecewise linear embedding of S^m in Q. D. B. A. Epstein suggested to me some time ago as a candidate for a counterexample to the Sphere Theorem in four dimensions an S^2 in E^4, which cuts itself orthogonally in just one point, fattened in E^4. This is essentially the manifold Q for $m=2$. The counterexample is proved by studying the universal covering space of Q.

COUNTEREXAMPLE 4. $M = B^m$, $\dot{M} = S^{m-1}$, $Q = S^1 \times B^{2m-1}$, $\dot{Q} = S^1 \times S^{2m-2}$, f: $M \to Q$ is a piecewise linear map such that $f(\dot{M}) \subset \dot{Q}$ links itself once homologically around the S^1 of $\dot{Q}$. There exists no piecewise linear embedding g of M in Q such that $f | \dot{M} = g | \dot{M}$. This follows from Counterexample 3, or may be proved independently by applying the Sublemma to the universal covering space of Q. It shows the necessity for the condition that $f(\dot{M})$ should not be contained in $\dot{Q}$ in Theorem 3.

REFERENCES

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