RESEARCH PROBLEMS

1. Frank Harary: *Matrix theory.*

What is the maximum value of the determinant among all square matrices of order n:

(a) with entries 0 or 1 (*binary matrices*)?

(b) with entries -1 or $+1$?

(c) with entries -1, 0, or $+1$?

At the International Symposium on Matrix Computation, April 24–28, 1961, the first question was asked by E. Aparo and the other two by L. Collatz. It was pointed out that the answer to (b) is known when $n=2^m$.

Classical Function Theory Problems

A Colloquium on Classical Function Theory was held at Cornell University August 17–21, 1961 with support from the National Science Foundation.

The following is a list of problems drawn up by members of the Colloquium. The list contains both old and new problems. The notations are those used in Nevanlinna's book "Eindeutige analytische Funktionen." (Received October 16, 1961.)

2. If $f(z)$ is entire and $\lim_{r\to\infty} \log M(r)r^{-\rho}$ exists, does $\lim T(r)r^{-\rho}$ exist?

3. Let $n_1(r, a)$ denote the number of simple zeros of $f(z)-a$ in $|z| \leqq r$. Conjecture: If $f(z)$ is an entire function of finite order ρ and $n_1(r, a)=O(r^c)$, $n_1(r, b)=O(r^c)\,(c<\rho)\,a\neq b$, $a, b\neq\infty$, then ρ is an integer multiple of $1/2$.

4. Functions satisfying $|f(z)| \leqq Me^{k|z|}$ in the plane and $|f(x)| \leqq A$ on the real axis are known to satisfy $|f(z)| \leqq Ae^{k|y|}$. What conclusion can be reached if the condition $|f(x)| \leqq A$ is replaced by $|f(x)| \leqq A$ on the positive real axis and $|f(x)| \leqq B$ on the negative axis? The real axis may be divided in various other ways and similar questions asked. What part does the constant M play? If $M<A$ we evidently have additional information.

5. Is there a bounded analytic function defined in $|z| <1$ such that $N(r,f')/(-\log(1-r))\to 1$ as $r\to 1$? Can such an example be constructed as a gap series $f(z)=\sum c_n z^{\lambda_n}$, $\sum |c_n| <\infty$?

6. If $f(z)$ is meromorphic in $|z| <1$ and $T(1,f')<\infty$, is $T(1,f)<\infty$?

7. If $f_1(z)$ and $f_2(z)$ are two entire functions of lower order less than one and if $f_1(z)$ and $f_2(z)$ have the same zeros, is $f_1(z)/f_2(z)$ a constant?

8. If $f(z)$ of finite order λ is entire and has a finite deficient value, find the best possible lower bound for the lower order μ of $f(z)$. It is conjectured that $\mu \geqq \lambda/2$ is the best estimate [Edrei and Fuchs, Trans. Amer. Math. Soc. **93** (1959), 292–328].

9. Prove or disprove: If $P(z)$ is a polynomial of degree n, then $|P'(z)/P(z)| < 1$ outside circles the sum of whose radii is less than An, where A is an absolute constant. This would have several consequences in the theory of meromorphic functions.

10. If $f(z)$ is an entire function of infinite order all of whose zeros are real, is $\delta(0, f) > 0$?

11. If $[f(z)]$ is a meromorphic function of finite order with more than two deficient values, is $\sup_r T(\sigma r)/T(r)$ finite for $\sigma > 1$?

12. If $\sum |a_n| < \infty$ and $f(z) = \sum a_n z^n$ is univalent in $|z| < 1$, can $f(z)$ map the unit circle on a curve of positive two-dimensional measure?

13. Corresponding to each function f analytic in the unit disk D and each value $w(|w| < 1)$, write

$$f_w(z) = f\left(\frac{z - w}{1 - \bar{w}z}\right) = \sum_0^\infty a_n^{(w)} z^n,$$

$$\|f_w\| = \sum_0^\infty |a_n^{(w)}|,$$

and let W_f be the set of all values $w(|w| < 1)$ for which $\|f_w\| < \infty$.

Since $a_n^{(w)}$ is a continuous function of w, W_f is a set of type F_σ. What more can be said? For example, if W_f is everywhere dense in D (or uncountable, or of positive measure), is W_f the unit disk? It is known that W_f may be a proper nonempty subset of D. Is W_f either empty or all of D if f is univalent?

14. Let $f(z)$ be regular in a simply connected domain D. It is known that $f(z)$ can be expanded in a series of Faber polynomials, $\sum_{n=0}^\infty a_n P_n(z)$. Find the domain of variability V of a_n as $f(z)$ runs through all functions regular in D and having positive real part there. It is known that if D is a circle V is a circle and if D is an ellipse then V is an ellipse.

15. Let G be a convex domain, let F be a relatively closed, totally

disconnected set in G, and let $B=G\backslash F$. Is it true that if f is defined in B and $\Re f'(z)>0$ in B, then f is univalent in B?

16. Let C_1 be a closed curve inside the unit circle C_2. Under what conditions on C_1 does there exist a univalent function f in the unit disk such that $f(C_1)$ and $f(C_2)$ are both convex?

17. If $f(z)=1/z+\sum_1^\infty a_n z^n$ is regular and univalent in $0<|z|<1$, find the best-possible estimate $a_n=O(n^{-\gamma})$. It is known that $\gamma<1$ and it has been conjectured that $\gamma=1/2$.

18. If $F(z)=z+\sum_2^\infty a_n z^n$ is typically real in $|z|<1$ then $F(z)=z/(1-z^2)P(z)$ where $P(0)=1$, $\Re P(z)>0$ in $|z|<1$. What other conditions must $P(z)$ satisfy to make $F(z)$ univalent in $|z|<1$?

19. If $f(z)=z+\sum_2^\infty a_n z^n$ is univalent and star-like of order 1/2 in $|z|<1$, i.e., $zf'(z)/f(z)\geqq 1/2$ for $|z|<1$, what is the radius of the largest circle $|z|<R$ within which $f(z)$ is convex? In other words when is

$$\min_{|z|=r} \Re\left[\frac{P+1}{2}+\frac{zP'(z)}{P+1}\right]>0$$

where $\Re P(z)>0$ in $|z|<1$, $P(0)=1$?

20. Let $f(z)=z+a_2z^2+\cdots$ be univalent in $|z|<1$. If $a_{n_k}=0$ for $n_k\to\infty$, $n_{k+1}-n_k=O(1)$, is it true that $a_n=O(1)$? More generally: Is it true that

$$|a_{n+1}|-|a_n|=O(1)?$$

(Biernacki proved $|a_{n+1}|-|a_n|=O((\log n)^{1/2})$ [Bull. Acad. Polon. Sci. Cl. III, **4** (1956), No. 1].)

21. Let $f(z)$ be regular and p-valent in $|z|<1, f(0)=0, f'(0)=1$, let $g(z)$ be regular and q-valent in $|z|<1$, $g(0)=0$, $g'(0)=1$. Is $f(z)+g(z)$ at most $(p+q)$-valent? Is the same true for $f(z)g(z)$? If f and g are convex and univalent, is it true that $f+g$ is starlike and univalent?

22. Let $R(p, k)=[2p-1-2(p^2-p)^{1/2}]^{1/k}$. Let $F(z)=z+\sum_{n=2}^\infty a_n z^n$ be regular in $|z|<1$, and without zeros in $0<|z|<1$. Conjecture: If $F(z)$ has k critical points in $|z|\leqq R(p, k)$, at least one of which is in $|z|<R(p, k)$, then $F(z)$ can not be p-valent. (It is known that there is a p-valent function with k critical points on $|z|=R(p, k)$.)

23. Find an extension of the Gehring-Lohwater generalization of Lindelöf's Theorem for Riemann Surfaces (See Math. Nachr. (1958), 165–170).

For the following problems see Proc. Nat. Acad. Sci. U.S.A. (1961), 98–105.

24. Let $y_1+iy_2=f(x_1+ix_2)$ be a quasi conformal mapping of the x_1x_2-plane onto the y_1y_2-plane. Can we find a quasi conformal mapping of the half space $x_3>0$ onto $y_3>0$ for which the given mapping is the boundary correspondence?

25. Let D be a domain in 3-space which can be mapped quasi conformally onto the unit sphere U and let $K(D)$ be the infimum of all K for which there exists a K-quasi conformal mapping of D onto U. Calculate $K(D)$ for D a cube, cylinder or any domain not a sphere or half space.

THE OCTOBER MEETING IN CAMBRIDGE

The five hundred eighty-third meeting of the American Mathematical Society was held in Cambridge, Massachusetts at the Massachusetts Institute of Technology on October 28, 1961. There were about 220 persons in attendance, including 190 registered members of the Society.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor Michel Kervaire of New York University addressed the Society on *Some results and problems in differential topology.* Professor Raoul Bott presided at the invited address and introduced the speaker. There were seven sessions for contributed papers, including a session for late papers, at which 47 papers were presented. The chairmen of these sessions were Professors A. E. Anderson, D. J. Benney, Bernard Epstein, J. G. Glimm, E. E. Moise, Hartley Rogers, Jr., and Dr. F. S. van Vleck.

EVERETT PITCHER,
Associate Secretary