SOLUTION OF THE DIRICHLET PROBLEM FOR EQUATIONS NOT NECESSARILY STRONGLY ELLIPTIC

BY MARTIN SCHECHTER

Communicated by E. Hille, August 7, 1958

Let $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ be a sequence of indices and set

$$|\mu| = \sum \mu_k, \quad D^\mu = \partial^{|\mu|}/(i\partial x_1)^{\mu_1}(i\partial x_2)^{\mu_2} \cdots (i\partial x_n)^{\mu_n},$$

$$\xi^\mu = \xi_1^{\mu_1}\xi_2^{\mu_2} \cdots \xi_n^{\mu_n}$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ is any n-dimensional vector. The linear partial differential operator

$$A = \sum_{|\mu| \leq m} a_\mu(x) D^\mu$$

with complex coefficients a_μ is elliptic at a point x if

$$P(x, \xi) \equiv \sum_{|\mu| = m} a_\mu(x)\xi^\mu \neq 0$$

for all real $\xi \neq 0$. It is strongly elliptic there if there is a complex constant γ such that $\operatorname{Re} \gamma P(x, \xi) \neq 0$ for $\xi \neq 0$. Let G be a bounded domain in n-space and let f and u_0 be smooth complex functions defined in G. The Dirichlet problem (A, f, u_0) is to find a complex function u such that $Au = f$ in G and all derivatives of $u - u_0$ of order $< m/2$ vanish on the boundary $\dot{G}$ of G. Gårding [2] and others have shown that if $\dot{G}$ and the coefficients a_μ are sufficiently smooth, a unique solution exists provided A is strongly elliptic and $a_{00\cdots 0}$ is large enough.

In this paper we extend the existence theory to include any elliptic operator for $n > 2$ and to operators satisfying a root condition [5] if $n = 2$. Such operators will be called properly elliptic. For $m = 2$ all properly elliptic operators are strongly elliptic, but this is not the case for higher orders. For example, the operator corresponding to

$$P(x, \xi) = \xi_1^4 + \xi_2^4 - \xi_3^4 + i(\xi_1^2 + \xi_2^2)\xi_3^2$$

is not strongly elliptic.

THEOREM. *Let A be properly elliptic and denote its formal adjoint by A^*. Assume that the Dirichlet problem $(A^*, 0, 0)$ has only the solution*

$u \equiv 0$. *Then for any f and u_0 sufficiently smooth the Dirichlet problem (A, f, u_0) has a solution.*

SKETCH OF PROOF. Without loss of generality, we may assume $u_0 \equiv 0$ and for convenience we assume $f \in C^\infty(\overline{G})$. Set

$$(v, w)_s = \sum_{|\mu| \leq s} \int_G D^\mu v \overline{D^\mu w} dx \qquad \|v\|_s^2 = (v, v)_s$$

and let V be the set of all $v \in C^\infty(\overline{G})$ having all derivatives of order $< m/2$ vanishing on $\dot{G}$. Complete V with respect to the norm $\| \ \|_m$ and call the resulting Hilbert space H. From the assumptions on A and A^* it follows [5] that

$$c^{-1}\|v\|_m \leqq \|A^*v\|_0 \leqq c\|v\|_m \qquad \text{for all } v \in H.$$

Hence, by the Lax-Milgram lemma [3] there is a $g \in H$ such that

$$(A^*g, A^*v)_0 = (f, v)_0 \qquad \text{for all } v \in H.$$

Applying the regularity theory of Nirenberg [4] and Browder [1], we see that $g \in C^\infty(\overline{G})$. Hence $AA^*g = f$ in G. Set $u = A^*g \in C^\infty(\overline{G})$. Then $Au = f$ in G and

$$(u, A^*v)_0 = (Au, v)_0 \qquad \text{for all } v \in H.$$

This last equality implies $u \in H$. The proof is thus complete.

The foregoing method can also be applied to systems of equations and to general boundary problems which cover A in the sense of [6].

REFERENCES

1. F. E. Browder, *On the regularity properties of solutions of elliptic differential equations*, Comm. Pure Appl. Math. vol. 9 (1956) pp. 351–361.

2. Lars Gårding, *Dirichlet's problem for linear elliptic partial differential equations*, Math. Scand. vol. 1 (1953) pp. 55–72.

3. P. D. Lax and A. N. Milgram, *Parabolic equations*, Annals of Mathematical Studies, no. 33, 1954, pp. 167–190.

4. Louis Nirenberg, *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math. vol. 8 (1955) pp. 648–674.

5. Martin Schechter, *On estimating elliptic partial differential operators in the L_2 norm*, Amer. J. Math. vol. 79 (1957) pp. 431–443.

6. ———, *Integral inequalities for partial differential operators and functions satisfying general boundary conditions*, Comm. Pure Appl. Math. to appear in 1959.

NEW YORK UNIVERSITY