ON THE EQUICONVERGENCE OF FOURIER SERIES AND FOURIER INTEGRALS

FÉLIX EDUARDO HERRERA

Given any function $f(x)$ defined for $-\infty < x < \infty$, let

$$(1) \qquad \int_0^\infty \{a(u) \cos ux + b(u) \sin ux\} du$$

be the Fourier integral (F.i.) of $f(x)$. Here

$$(2) \qquad a(u) = \frac{1}{\pi} \int_{-\infty}^\infty f(t) \cos utdt; \qquad b(u) = \frac{1}{\pi} \int_{-\infty}^\infty f(t) \sin utdt$$

are the cosine and sine transforms of $f(x)$. They exist as absolutely convergent integrals if $f \in L(-\infty, \infty)$, or in a certain generalized sense, if $f \in L^p(-\infty, \infty)$ with $1 < p \le 2$.

In all these cases, the partial integrals of (1) are given by the formula

$$S_\omega(x, f) = \int_0^\omega \{a(u) \cos ux + b(u) \sin ux\} du$$

$$= \frac{1}{\pi} \int_{-\infty}^\infty f(x+t) \frac{\sin \omega t}{t} dt,$$

but the last integral has meaning if

$$(3) \qquad \int_{-\infty}^\infty \frac{|f(t)|}{1 + |t|} dt < \infty$$

even if the transforms (2) do not exist.

Now, it is well known that the problem of the representation of $f(x)$ as the limit of

$$(3a) \qquad S_\omega(x, f) = \frac{1}{\pi} \int_{-\infty}^\infty f(x+t) \frac{\sin \omega t}{t} dt$$

for $\omega \to \infty$ can be reduced to the representation of $f(x)$ by a Fourier series (F.s.). More precisely, we have the following theorem:

(A) *Given any function satisfying* (3), *and any interval* $I_a = (a, a+2\pi)$ *of length* 2π, *let* $f_a(x)$ *be the function of period* 2π *and coinciding with* $f(x)$ *in* I_a. *Let*

---

(4)
$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the F.s. of f(x) and

$$s_\omega(x, f_a) = \frac{1}{2} a_0 + \sum_{n \leq \omega} (a_n \cos nx + b_n \sin nx), \qquad \omega \geq 0,$$

the partial sums of (4). *Then the difference*

(5)
$$\Delta_\omega(x) = \Delta_\omega(x, f, a) = S_\omega(x, f) - s_\omega(x, f_a)$$

tends to zero at every point inside I_a, and the convergence is uniform in every closed interval interior to I_a.[1]

Thus, tests of convergence which are valid for F.s. are immediately extensible to F.i. Moreover, since the convergence of (5) to zero implies its summability (C, α) for any $\alpha > 0$, we immediately obtain for the summability of F.i. results analogous to those for F.s.

Now, let

(6)
$$\int_0^\infty \{a(u) \sin ux - b(u) \cos ux\} du$$

be the integral conjugate to (1). The partial integrals of (6) can be written

(6a)
$$\tilde{S}_\omega(x, f) = -\frac{1}{\pi} \int_{-\infty}^\infty f(x + t) \frac{1 - \cos \omega t}{t} dt$$

if $f \in L^p(-\infty, \infty)$ with $1 \leq p \leq 2$, but again the right-hand side here has meaning only if (3) is satisfied. The following result completes (A):

(B) *Let*

$$\tilde{s}_\omega(x, f_a) = \sum_{n \leq \omega} (a_n \sin nx - b_n \cos nx), \qquad \omega \geq 1.$$

On the assumptions of (A), *the difference*

$$\tilde{\Delta}_\omega(x) = \tilde{\Delta}_\omega(x, f, a) = \tilde{S}_\omega(x, f) - \tilde{s}_\omega(x, f_a)$$

tends to a limit (not necessarily zero) at every point x interior to I_a, and the convergence is uniform over any closed interval interior to I_a.[2]

---

[1] See Zygmund, *Trigonometrical series*, p. 306. The book will henceforth be quoted T.S.

[2] Pi Calleja, P., *Über die Konvergenzbedingungen der komplexen Form des Fourierschen Integralen*, Math. Zeit. vol. 40 (1935–1936).

In a number of problems of the theory of F.s., we have to consider the behaviour of a F.s. differentiated term by term k times. A typical result in this order of ideas is the following:

(C) *Let $f_{(k)}(x)$ be the kth generalized derivative of $f(x)$. If $f_{(k)}(x_0)$ exists, the F.s. of $f(x)$, differentiated term by term k times, is, at the point x_0, summable (C, α), $\alpha > k$, to the value $f_{(k)}(x_0)$.*[3]

Similar problems exist for F.i.

In the present paper we prove a theorem that enables us to reduce a F.i. problem of the aforementioned kind to the corresponding problem on F.s., namely:

THEOREM. *Let $f(x)$ satisfy condition* (3). *Let $I_a = (a, a+2\pi)$ and let $f_a(x)$ be the function of period 2π coinciding with $f(x)$ in I_a. Let $S_\omega(x, f)$ be the partial integral* (3a) *and let $s_\omega(x, f_a)$ be the partial sums of the F.s. of $f_a(x)$. Let $k = 1, 2, \cdots$. Then the difference*

$$\frac{d^k}{dx^k} S_\omega(x, f) - \frac{d^k}{dx^k} s_\omega(x, f_a)$$

is uniformly summable (C, k) to zero in every closed interval interior to I_a.

Similarly, the difference

$$\frac{d^k}{dx^k} \tilde{S}_\omega(x, f) - \frac{d^k}{dx^k} \tilde{s}_\omega(x, f_a)$$

is uniformly summable (C, k) in every closed interval interior to I_a.

This theorem complements the results (A) and (B) stated above and is reduced to these results when $k = 0$. An immediate application of it leads us, when (C) is taken into account, to:

(C') *If $f_{(k)}(x_0)$ exists, the F.i. of $f(x)$, differentiated k times under the integral sign, is, at the point x_0, summable (C, α), $\alpha > k$, to the value $f_{(k)}(x_0)$.*

In this connection we may recall the definition of summability of a series by M. Riesz's method of order α or summability (R, α) and the fact that if a series is summable to sum s by the method (R, α), it is also summable to the same sum by the method (C, α) and conversely.[4]

We shall base the proof of the theorem on the following lemmas:

(i) *Let $g(t)$ be an integrable function in $(-\pi, \pi)$ and of period 2π. Let $r > 1$ and $0 < \delta < \pi$. Then the integral*

---

[3] T.S. p. 257.

[4] T.S. p. 312.

$$\int_{|t| \geq \pi} \frac{g(t)}{(x - t)^r} \, dt$$

converges absolutely and uniformly for $-\pi + \delta \leq x \leq \pi - \delta$.

The proof of this lemma is immediate.

(ii) *Let* $0 < \eta < 1/2$. *Under the conditions of Lemma* (i), *the integral*

$$\left\{ \int_{\pi}^{\infty} + \int_{-\infty}^{-\pi} \right\} \frac{g(t)}{x - t} e^{i\omega(x-t)} dt$$

$$= \lim_{T \to \infty} \left\{ \int_{\pi}^{T} + \int_{-T}^{-\pi} \right\} \frac{g(t)}{x - t} e^{i\omega(x-t)} dt$$

converges:

  (a) *uniformly for* $-\pi + \delta \leq x \leq \pi - \delta$ *and any fixed* ω,

  (b) *uniformly for* $-\pi + \delta \leq x \leq \pi - \delta$, $\eta \leq \omega \leq 1 - \eta$,

  (c) *boundedly for* $-\pi + \delta \leq x \leq \pi - \delta$ *and any range of variation of* ω.

Clearly it is sufficient to suppose that $T = (2n+1)\pi$. On such an assumption we have

$$\left\{ \int_{\pi}^{T} + \int_{-T}^{-\pi} \right\} \frac{g(t)}{x - t} e^{i\omega(x-t)} dt$$

$$= \sum_{k=1}^{n} \int_{-\pi}^{\pi} g(t) e^{i\omega(x-t)} \left\{ \frac{e^{2k\pi i\omega}}{2k\pi + (x - t)} - \frac{e^{-2k\pi i\omega}}{2k\pi - (x - t)} \right\} dt.$$

By direct inspection of the right-hand side of this formula we get (a) in the special case $\omega = $ an integer. Now, let ω be non-integral. Summation by parts gives

$$\sum_{k=1}^{\infty} \frac{e^{\pm 2k\pi i\omega}}{2k\pi \pm (x - t)}$$

$$= \sum_{k=1}^{\infty} \left\{ \sum_{\nu=1}^{k} e^{\pm 2\nu\pi i\omega} \right\} \left\{ \frac{1}{2k\pi \pm (x - t)} - \frac{1}{2(k+1)\pi \pm (x - t)} \right\}$$

$$= \frac{2\pi e^{\pm 2\pi i\omega}}{1 - e^{\pm 2\pi i\omega}} \sum_{k=1}^{\infty} \frac{1 - e^{\pm 2k\pi i\omega}}{\{2k\pi \pm (x - t)\}\{2(k+1)\pi \pm (x - t)\}}$$

so that

$$\left| \sum_{k=1}^{\infty} \frac{e^{\pm 2k\pi i\omega}}{2k\pi \pm (x - t)} \right| < \frac{1}{|1 - e^{\pm 2\pi i\omega}|} \left(\frac{2}{\delta} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \right),$$

and from this result (a) and (b) follow.

To obtain (c), let us consider the expression

$$\sum_{k=1}^{n} \left\{ \frac{e^{2k\pi i\omega}}{2k\pi + (x-t)} - \frac{e^{-2k\pi i\omega}}{2k\pi - (x-t)} \right\}$$

$$= 2i \sum_{k=1}^{n} \frac{\sin 2k\pi\omega}{2k\pi - (x-t)^2/2k\pi} - 2(x-t) \sum_{k=1}^{n} \frac{\cos 2k\pi\omega}{4k^2\pi^2 - (x-t)^2}$$

$$= 2iP_n(\omega, x-t) - 2(x-t)Q_n(\omega, x-t).$$

Obviously $\lim_{n\to\infty} Q_n(\omega, x-t)$ exists uniformly for $-\pi+\delta \leq x \leq \pi-\delta$ and any range of variation of ω. Furthermore the difference

$$P_n(\omega, x-t) - \sum_{k=1}^{n} \frac{\sin 2k\pi\omega}{2k\pi}$$

$$= (x-t)^2 \sum_{k=1}^{n} \frac{\sin 2k\pi\omega}{(2k\pi)^2 \{2k\pi - (x-t)^2/2k\pi\}},$$

as $n \to \infty$, converges also uniformly for $-\pi+\delta \leq x \leq \pi-\delta$ and any range of variation of ω. Since it is known that the series $\sum_{k=1}^{\infty} \sin 2k\pi\omega/2k\pi$ is boundedly convergent in any interval of ω, (c) is proved.

(iii) *Let $v \neq 0$. Then for $q=1, 2, \cdots; k=1, 2, \cdots$,*

$$\frac{d^q}{dv^q} \int_0^{\Omega} \left(1 - \frac{\omega}{\Omega}\right)^k e^{i\omega v} d\omega$$

$$= (-1)^q \frac{q!}{v^{q+1}} i + \sum_{\nu=0}^{k-2} \frac{(-1)^q k(k-1) \cdots (k-\nu)(\nu+1+q)!}{i^\nu(\nu+1)!\Omega^{\nu+1}v^{\nu+q+2}}$$

(7)

$$+ \sum_{\nu=0}^{q-1} \frac{(-1)^\nu C_{q,\nu}(k+\nu)!e^{i\Omega v}}{i^{(k+\nu+1)-q}\Omega^{k+\nu-q}v^{k+\nu+1}} + \frac{(-1)^q(k+q)!}{i^{k+1}\Omega^k v^{k+q+1}} (e^{i\Omega v} - 1)$$

where we put $\sum_{\nu=0}^{-1} \psi(\nu) = 0$.[5]

We now need to prove that

$$\int_0^{\Omega} \left(1 - \frac{\omega}{\Omega}\right)^k e^{i\omega v} d\omega = -\frac{1}{iv} - \sum_{\nu=0}^{k-2} \frac{k(k-1) \cdots (k-\nu)}{\Omega^{\nu+1}(iv)^{\nu+2}}$$

(8)

$$+ \frac{k!}{\Omega^k} \frac{e^{i\Omega v} - 1}{(iv)^{k+1}},$$

because by differentiation of this formula with respect to v we obtain the preceding one.

Now, (8) is certainly true for $k=1$. Let us assume that it is true for

---

[5] $\psi(\nu)$ stands in general for any function of the non-negative integer ν appearing in this paper.

$k = n$. We shall prove that on this hypothesis the formula holds also true for $k = n+1$. Indeed,

$$\int_0^\Omega \left(1 - \frac{\omega}{\Omega}\right)^{n+1} e^{i\omega v} d\omega$$

$$= \left[\left(1 - \frac{\omega}{\Omega}\right)^{n+1} \frac{e^{i\omega v}}{iv}\right]_0^\Omega + \frac{n+1}{iv\Omega} \int_0^\Omega \left(1 - \frac{\omega}{\Omega}\right)^n e^{i\omega v} d\omega$$

$$= -\frac{1}{iv} - \sum_{\nu=0}^{n-1} \frac{(n+1)n \cdots (n+1-\nu)}{\Omega^{\nu+1}(iv)^{\nu+2}} + \frac{(n+1)!}{\Omega^{n+1}} \frac{e^{i\Omega v} - 1}{(iv)^{n+2}},$$

which was to be proved.

In particular, if $q = k$ and A_ν, B_ν, and C are constants, (7) can be written in the form

$$(9) \qquad \frac{d^k}{dv^k} \int_0^\Omega \left(1 - \frac{\omega}{\Omega}\right)^k e^{i\omega v} d\omega = (-1)^k \frac{k!}{v^{k+1}} i + \Phi_\Omega(k, v)$$

where

$$(10) \qquad \Phi_\Omega(k, v) = \sum_{\nu=0}^{k-2} \frac{A_\nu}{\Omega^{\nu+1} v^{\nu+k+2}} + \sum_{\nu=0}^{k-1} \frac{B_\nu}{\Omega^\nu v^{\nu+k+1}} e^{i\Omega v}$$
$$+ \frac{C}{\Omega^k v^{2k+1}} (e^{i\Omega v} - 1).$$

These lemmas having been proved, we pass to the proof of the theorem. For the sake of simplicity, let us assume $a = 0$ and let us write f_0, J_0, J_0' instead of f_a, J_a, J_a'. Moreover let us put

$$\overset{*}{s}_\omega(x) = \overset{*}{s}_\omega(x, f_0) = \frac{a_0}{2} + \sum_{n \leq \omega}'(a_n \cos nx + b_n \sin nx), \quad \omega \geq 0,$$

where the dash $'$ on the right indicates that if ω is an integer, we must multiply the last term by one-half. Thus, for ω positive and not integral,

$$\overset{*}{s}_\omega(x) = s_\omega(x),$$

and therefore instead of considering the (C, α) means of the difference $S_\omega(x) - s_\omega(x)$, we may consider the (C, α) means of the difference $S_\omega(x) - \overset{*}{s}_\omega(x)$.

Now, it is well known that if $g(t)$ is any periodic function of period 2π, integrable in the interval $(0, 2\pi)$, then[6]

---

[6] G. H. Hardy, *Notes on some points in the integral calculus*, Messenger of Mathematics vol. 52 (1922) pp. 49–53.

$$s_\omega^*(x, g) = \frac{1}{\pi} \int_{-\infty}^{\infty} g(t) \frac{\sin \omega(x - t)}{x - t} \, dt = \lim_{T \to \infty} \frac{1}{\pi} \int_{-T}^{T} g(t) \frac{\sin \omega(x - t)}{x - t} dt.$$

Consequently, the first part of the theorem will be proved if we prove that, as $\Omega \to \infty$,

$$\frac{k}{\pi \Omega^k} \int_0^\Omega (\Omega - \omega)^{k-1} d\omega \frac{d^k}{dx^k} \int_{-\infty}^{\infty} \{f(t) - f_0(t)\} \frac{\sin \omega(x - t)}{x - t} \, dt \to 0$$

uniformly in J_0'.

For that purpose, let $h(t) = f(t) - f_0(t)$, so that $h(t) = 0$ in $(-\pi, \pi)$, and let us consider the integral

$$J_\Omega(k, x) = \frac{k}{\pi \Omega^k} \int_0^\Omega (\Omega - \omega)^{k-1} d\omega \frac{d^k}{dx^k} \int_{|t| \geq \pi} h(t) \frac{\sin \omega(x - t)}{x - t} \, dt.$$

In virtue of assumption (3) and lemmas (i) and (ii), the integral

$$\int_{|t| \geq \pi} h(t) \left(\frac{d^k}{dx^k} \frac{e^{i\omega(x-t)}}{x - t} \right) dt$$

is, for every fixed ω, uniformly convergent in J_0', so that we may write, $x \in J_0'$,

$$J_\Omega(k, x) = \frac{k}{\pi \Omega^k} \int_0^\Omega (\Omega - \omega)^{k-1} d\omega \int_{|t| \geq \pi} h(t) \left(\frac{d^k}{dx^k} \frac{\sin \omega(x - t)}{x - t} \right) dt$$

and also

$$J_\Omega(k, x) = \frac{1}{\pi} \int_{|t| \geq \pi} h(t) dt \frac{k}{\Omega^k} \int_0^\Omega (\Omega - \omega)^{k-1} \left(\frac{d^k}{dx^k} \frac{\sin \omega(x - t)}{x - t} \right) d\omega,$$

the inversion of integrations being justified by (3) and lemmas (i) and (ii).

Integrating the inner integral by parts, we get

$$J_\Omega(k, x) = \frac{1}{\pi} \int_{|t| \geq \pi} h(t) dt \int_0^\Omega \left(1 - \frac{\omega}{\Omega} \right)^k \left(\frac{d^k}{dx^k} \cos \omega(x - t) \right) d\omega$$

$$= \frac{1}{\pi} \int_{|t| \geq \pi} h(t) dt \frac{d^k}{dx^k} \int_0^\Omega \left(1 - \frac{\omega}{\Omega} \right)^k \cos \omega(x - t) d\omega$$

$$= \frac{1}{\pi} \Re \left\{ \int_{|t| \geq \pi} h(t) dt \frac{d^k}{dx^k} \int_0^\Omega \left(1 - \frac{\omega}{\Omega} \right)^k e^{i\omega(x-t)} d\omega \right\}$$

or, according to (9),

$$J_\Omega(k, x) = \frac{1}{\pi} \Re \left\{ \int_{|t| \geq \pi} h(t) \Phi_\Omega(k, x - t) dt \right\}.$$

Let us consider the integral $\int_{|t| \geq \pi} h(t) \Phi_\Omega(k, x-t) dt$ and let us suppose we have replaced $\Phi_\Omega(k, x-t)$ by its expression (10). If so, we shall obtain integrals belonging to any one of the following types:

$$\frac{1}{\Omega^s} \int_{|t| \geq \pi} \frac{h(t)}{(x - t)^r} dt; \qquad \frac{1}{\Omega^s} \int_{|t| \geq \pi} \frac{h(t)}{(x - t)^r} e^{i\Omega(x-t)} dt;$$

$$\int_{|t| \geq \pi} \frac{h(t)}{(x - t)^2} e^{i\Omega(x-t)} dt$$

where s and r are positive integers such that $s \geq 1$, $r \geq 3$.

Clearly every integral belonging to either of the two first types tends to zero uniformly in J_0' as $\Omega \to \infty$. As for $\int_{|t| \geq \pi} (h(t)/(x-t)^2) e^{i\Omega(x-t)} dt$, we have

$$\int_{|t| \geq \pi} \frac{h(t)}{(x - t)^2} e^{i\Omega(x-t)} dt = \left\{ \int_{A \geq |t| \geq \pi} + \int_{|t| \geq A} \right\} \frac{h(t)}{(x - t)^2} e^{i\Omega(x-t)} dt$$

$$= I_1 + I_2.$$

If we choose A large enough, by (3) and Lemma (i), $|I_2| < \epsilon/2$ for any $x \in J_0'$. On the other hand, $1/(x-t)^2$ being for $x \in J_0'$, $\pi \leq |t| \leq A$ a uniformly bounded and uniformly continuous function depending on the parameter x, $I_1 \to 0$ as $\Omega \to \infty$ uniformly in J_0', so that

$$\int_{|t| \geq \pi} \frac{h(t)}{(x - t)^2} e^{i\Omega(x-t)} dt \to 0$$

uniformly in J_0' and the first part of the theorem is proved. For the second part, we begin by putting

$$\overset{*}{s}_\omega(x) = \overset{*}{s}_\omega(x, f_0) = \sum_{n \leq \omega}' (a_n \sin nx - b_n \cos nx), \qquad \omega \geq 1,$$

where the dash $'$ on the right has the same meaning as before. Now, by analogy with $s_\omega^*(x)$, the expression $\overset{*}{s}_\omega(x)$ can be written in the form

$$\overset{*}{s}_\omega(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} f_0(t) \frac{1 - \cos \omega(x - t)}{x - t} dt$$

$$= -\lim \frac{1}{\pi} \int_{-T}^{T} f_0(t) \frac{1 - \cos \omega(x - t)}{x - t} dt,$$

and the device used in the first part of the theorem entitles us to write

$$\frac{k}{\pi \Omega^k} \int_0^\Omega (\Omega - \omega)^{k-1} \frac{d^k}{dx^k} (\tilde{S}_\omega(x) - \tilde{s}_\omega(x)) d\omega$$

$$= -\frac{1}{\pi} \Im \left\{ \int_{|t| \geq \pi} h(t) dt \frac{d^k}{dx^k} \int_0^\Omega \left(1 - \frac{\omega}{\Omega}\right)^k e^{i\omega(x-t)} d\omega \right\}$$

$$= \frac{(-1)^{k+1} k!}{\pi} \int_{|t| \geq \pi} \frac{h(t)}{(x-t)^{k+1}} dt$$

$$- \frac{1}{\pi} \Im \left\{ \int_{|t| \geq \pi} (t) \Phi_\Omega(k, x-t) dt \right\},$$

and inasmuch as $\int_{|t| \geq \pi} h(t) \Phi_\Omega(k, x-t) dt \to 0$ uniformly in J_0', we obtain the result

$$\lim_{\Omega \to \infty} \frac{k}{\pi \Omega^k} \int_0^\Omega (\Omega - \omega)^{k-1} \frac{d^k}{dx^k} (\tilde{S}_\omega(x) - \tilde{s}_\omega(x)) d\omega$$

$$= \frac{(-1)^{k+1} k!}{\pi} \int_{|t| \geq \pi} \frac{h(t)}{(x-t)^{k+1}} dt$$

uniformly in J_0', and this completes the proof of the theorem. Before ending, the author wishes to express to Professor A. Zygmund his gratitude for the assistance and encouragement shown to him while he was engaged on the present paper during his tenure of a scholarship at the University of Pennsylvania in 1946–1947.

UNIVERSITY OF TUCUMÁN AND
    UNIVERSITY OF PENNSYLVANIA