SOME REMARKS ON RULED SURFACES

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In a previous paper [1][1] the author showed that the projective differential geometry of a nondevelopable ruled surface S in three-dimensional space could be studied by means of the expansion

$$
\begin{aligned}
z = {} & xy - \frac{1}{3}\gamma y^3 - \frac{1}{3}\gamma_u xy^3 - \frac{1}{12}\gamma_v y^4 - \frac{1}{6}\gamma_{uu}x^2y^3 - \frac{1}{12}\gamma_{uv}xy^4 \\
& + \frac{1}{60}(10\gamma\gamma_u - \gamma_{vv})y^5 - \frac{1}{24}\gamma_{uuv}x^2y^4 \\
& + \frac{1}{60}(10\gamma\gamma_{uu} + 10\gamma_u^2 - \gamma_{uvv})xy^5 \\
& + \frac{1}{360}(20\gamma_u\gamma_v + 15\gamma\gamma_{uv} - \gamma_{vvv})y^6 + \cdots,
\end{aligned} \tag{1}
$$

for one nonhomogeneous coordinate z as a power series in the other two nonhomogeneous coordinates x and y. Here γ is a function of the form $A(v)u^2+B(v)u+C(v)$. It was also shown that there is a one-parameter family of cubic surfaces with fifth order contact with S, namely,

$$
\begin{aligned}
& \frac{1}{3}\gamma y^3 + Dz^3 + (z - xy)\left(Px - \frac{1}{4}(\gamma_v/\gamma)y + Mz + 1\right) \\
& \qquad + yz(Iy + Jz) = 0,
\end{aligned} \tag{2}
$$

where D is the parameter and P, M, I, and J are defined as follows:

$$
\begin{aligned}
P &= (15\gamma_v^2 + 40\gamma^2\gamma_u - 12\gamma\gamma_{vv})/80\gamma^3, \\
M &= (40\gamma^2\gamma_u\gamma_v + 12\gamma\gamma_v\gamma_{vv} - 80\gamma^3\gamma_{uv} - 15\gamma_v^3)/320\gamma^4, \\
I &= (5\gamma_v^2 + 40\gamma^2\gamma_u - 4\gamma\gamma_{vv})/80\gamma^2, \\
J &= (15\gamma_u\gamma_v^2 + 40\gamma^2\gamma_u^2 - 12\gamma\gamma_u\gamma_{vv} + 40\gamma^3\gamma_{uu})/240\gamma^3.
\end{aligned}
$$

In this paper we shall report some further results on nondevelopable ruled surfaces which can be obtained with the help of the above formulas.

It was shown in [1] that S is a cubic surface if and only if $\mathcal{A} = \mathcal{B} = 0$, where

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$\mathcal{A} = [(15\gamma_v^2 - 12\gamma\gamma_{vv} - 40\gamma^2\gamma_u)(\gamma_u\gamma_v - \gamma\gamma_{uv}) + 40\gamma^3(\gamma_u\gamma_v - \gamma\gamma_{uv})_u]/960\gamma^4,$$

$$\mathcal{B} = [1600\gamma^4(2\gamma\gamma_{uu} - \gamma_u^2) + 960\gamma^3(\gamma_u\gamma_v - \gamma\gamma_{uv})_v - 2400\gamma^2\gamma_v(\gamma_u\gamma_v - \gamma\gamma_{uv}) + (12\gamma\gamma_{vv} - 15\gamma_v^2)^2]/57600\gamma^4.$$

Since this criterion requires derivatives of γ of the third order, it is of interest to notice that the following theorem is true.

THEOREM. *The nondevelopable ruled surface* (1) *is cubic if and only if* $P=M=J=0$. *It is Cayley's cubic scroll if and only if* $I=0$ *also.*

If S is cubic we have seen [1] that we can assume that either $\gamma=3$ or $\gamma=3u/10v^2$. By direct substitution it is found in the first case, which corresponds to Cayley's cubic scroll, that $P=M=J=I=0$, and in the second case that $P=M=J=0$, $I=1/10v^2$. The converse follows from the identities

$$960\gamma^3\mathcal{A} = (15\gamma_v^2 - 12\gamma\gamma_{vv})(4\gamma M + \gamma_v P) + 40\gamma^3(4\gamma M + \gamma_v P)_u,$$

$$360\gamma\mathcal{B} = 40\gamma^2(3J - 2\gamma_u P + \gamma P^2) + 6\gamma(4\gamma M + \gamma_v P)_v - 9\gamma_v(4\gamma M + \gamma_v P),$$

which imply that $\mathcal{A}=\mathcal{B}=0$ if $M=P=J=0$.

If S is cubic, the surface (2) will actually have sixth order contact with S if and only if $D=P\gamma_{uu}/6$, and it will then coincide with S.

Let us now seek to find the double points of the cubic surface (2). The homogeneous coordinates of such a point must satisfy the equations $F_i=0$, where

$$F_1 = x_4\left(Px_2 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Mx_4 + x_1\right) + (x_1x_4 - x_2x_3),$$

$$F_2 = -x_3\left(Px_2 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Mx_4 + x_1\right) + P(x_1x_4 - x_2x_3),$$

$$F_3 = \gamma x_3^2 - x_2\left(Px_2 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Mx_4 + x_1\right) - \frac{1}{4}(\gamma_v/\gamma)(x_1x_4 - x_2x_3) + 2Ix_3x_4 + Jx_4^2,$$

$$F_4 = 3Dx_4^2 + x_1\left(Px_2 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Mx_4 + x_1\right) + M(x_1x_4 - x_2x_3) + Ix_3^2 + 2Jx_3x_4.$$

Setting $x_4=0$, we readily find that $(-P, 1, 0, 0)$ is the only double point in the tangent plane, and that this point is a double point for all the cubic surfaces (2).

To find double points not in the tangent plane, suppose that $x_4 \neq 0$. Then $F_1=F_2=0$ if and only if either

$$(3) \qquad Px_2 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Mx_4 + x_1 = 0, \quad x_1x_4 - x_2x_3 = 0,$$

or

$$(4) \qquad F_1 = 0, \qquad Px_4 + x_3 = 0.$$

When (3) holds we may set $x_4=1$ and see that a double point must satisfy (3) and

$$(5) \qquad \gamma x_3^2 + 2Ix_3 + J = 0, \qquad Ix_3^2 + 2Jx_3 + 3D = 0.$$

Let us suppose for the moment that $I^2 \neq \gamma J$. Then two values of x_3 are determined by the first of equations (5). Each value of x_3 determines a value of D by the second of equations (5). These values of D are different since $I^2 \neq \gamma J$. For each value of x_3 we determine x_1 and x_2 by solving the equations

$$(6) \qquad x_1 + Px_2 = \frac{1}{4}(\gamma_v/\gamma)x_3 - M, \qquad x_1 - x_2x_3 = 0.$$

Hence a unique x_1 and x_2 can be found unless $x_3=-P$. We conclude that if the surface S is such that $I^2 \neq \gamma J$, $\gamma P^2 - 2IP + J \neq 0$, then there are two cubic surfaces (2) each of which has a (unique) double point which is not in the tangent plane and satisfies equations (3).

On the other hand, if $\gamma P^2 - 2IP + J = 0$ and $I^2 \neq \gamma J$, then one root x_3 of the first of equations (5) is equal to $-P$ and the other is not. Corresponding to the root which is different from $-P$ there exists one cubic surface (2) which has a unique double point not in the tangent plane which satisfies equations (3). When $x_3=-P$, equations (3.4) have a solution if and only if

$$(7) \qquad 0 = M + \frac{1}{4}(\gamma_v/\gamma)P = (\gamma_u\gamma_v - \gamma\gamma_{uv})/4\gamma^2.$$

Since it is easy to see that

$$(8) \qquad \begin{aligned} 19200\gamma^5(\gamma P^2 - 2IP + J) = {} & (15\gamma_v^2 - 12\gamma\gamma_{vv})^2 \\ & - 1600\gamma^4(\gamma_u^2 - 2\gamma\gamma_{uu}), \end{aligned}$$

it is clear that when $\gamma P^2-2IP+J=0$ and $x_3=-P$, equations (3.4) have a solution if and only if $\mathcal{A}=\mathcal{B}=0$. In this case S is a cubic surface itself and the entire line $x_1+Px_2=x_3+Px_4=0$ consists of double points for the cubic surface (2) determined by the parameter $D=(-IP^2+2JP)/3$. Since S is cubic, $P=0$. Therefore, $D=0$ and hence the cubic surface (2) is the surface S itself. Since $I^2\neq\gamma J=0$, S is not Cayley's cubic scroll. We conclude that if $\gamma P^2-2IP+J=0$, $I^2\neq\gamma J$, then there is one cubic surface (2) which has a (unique) double point which is not in the tangent plane and satisfies equations (3). There is another cubic surface (2) which has a double point not in the tangent plane which satisfies equations (3) if and only if S is itself a cubic surface, not Cayley's cubic scroll, in which case the cubic surface (2) coincides with S which has the whole line $x_1=x_3=0$ as a line of double points.

When $I^2=\gamma J$, then only one value of x_3 is determined by the first of equations (5). If also $\gamma P^2-2IP+J\neq 0$, then this value of x_3 is different from $-P$ and so there exists one cubic surface (2) with a (unique) double point which is not in the tangent plane and satisfies equations (3).

When $I^2=\gamma J$ and $\gamma P^2-2IP+J=0$, reasoning like that used two paragraphs earlier shows that there is a cubic surface (2) with a double point not in the tangent plane which satisfies equations (3) if and only if S is Cayley's cubic scroll, in which case the cubic surface (2) is S itself and has the whole line $x_1=x_3=0$ as a line of double points.

Now let us consider double points which satisfy equations (4). Setting $x_3=-P$, $x_4=1$, we see that the equations $F_1=F_3=F_4=0$ become

$$(9)\qquad 2(x_1+Px_2) = -M-\frac{1}{4}(\gamma_v/\gamma)P = -(\gamma_u\gamma_v-\gamma\gamma_{uv})/4\gamma^2,$$

$$(10)\qquad (4\gamma x_2-\gamma_v)(\gamma_u\gamma_v-\gamma\gamma_{uv}) = 32\gamma^3(\gamma P^2-2IP+J),$$

$$(11)\qquad D = -(x_1-M)(\gamma_u\gamma_v-\gamma\gamma_{uv})/24\gamma^2-(IP^2-2JP)/3.$$

Unique values for x_1, x_2, and D can thus be determined provided that $\gamma_u\gamma_v-\gamma\gamma_{uv}\neq 0$. The cubic surface (2) so determined may, however, coincide with a cubic surface previously determined which has a double point satisfying equations (3). To find the condition that this occurs we explicitly solve equations (9), (10), and (11). Let $\eta=\gamma_u\gamma_v-\gamma\gamma_{uv}$, $\theta=5\gamma_v^2-4\gamma\gamma_{vv}$, $\Delta=\gamma_u^2-2\gamma\gamma_{uu}$. It follows from equations (10) and (8) that

$$x_2 = \gamma_v/4\gamma+3\theta^2/800\gamma^3\eta-2\gamma\Delta/3\eta$$

if $\eta \neq 0$. From (9) and the definitions of P and M we then find that

$$M - x_1 = 9\theta^3/64000\gamma^6\eta + 3\theta^2\gamma_u/1600\gamma^4\eta - \Delta\theta/40\gamma^2\eta$$
$$+ 3\eta/8\gamma^2 - \gamma_u\Delta/3\eta,$$

and finally we conclude from equation (11) and the definitions of I, P, and J that

$$3D = (6\theta\gamma\gamma_{uu} + 15\eta^2 + 80\gamma^3\gamma_u\gamma_{uu})/320\gamma^4. \tag{12}$$

The cubic surface (2) with this value of D coincides with a cubic surface (2) with a double point which is not in the tangent plane and satisfies equations (3) if and only if there is a value of x_3 which satisfies equations (5) simultaneously when D satisfies equation (12) and which belongs to a double point obtained from equations (3). Eliminating x_3 between equations (5) we see that this coincidence will occur if and only if

$$(3D\gamma - IJ)^2 - 4(I^2 - \gamma J)(J^2 - 3ID) = 0,$$

an equation which can be reduced in a straightforward manner to

$$\begin{aligned} - 81\Delta\theta^4 + 405\eta^2\theta^3 + 28800\gamma^4\Delta^2\theta^2 - 648000\gamma^4\eta^2\Delta\theta \\ + 2430000\gamma^4\eta^4 - 2560000\gamma^8\Delta^3 = 0. \end{aligned} \tag{13}$$

We conclude that if $\eta \neq 0$ and equation (13) fails to hold, then there is another cubic surface (2) in addition to the ones already found which has a double point not in the tangent plane.

If $\eta \neq 0$ and equation (13) holds, then there is a value of x_3 which satisfies equations (5) when D has the value given by equation (12). This value of x_3 is equal to $-P$ if and only if $\gamma P^2 - 2IP + J = 0$ and $3D = -IP^2 + 2JP$, and hence if and only if $M - x_1 = 0$ and $9\theta^2 = 1600\gamma^4\Delta$. When $9\theta^2 = 1600\gamma^4\Delta$ it is easy to see that $M - x_1 = 3\eta/8\gamma^2 \neq 0$. Hence the value of x_3 thus obtained can never be equal to $-P$. It follows that if either $I^2 - \gamma J$ or $\gamma P^2 - 2IP + J$ is different from zero no new cubic surfaces (2) having a double point not in the tangent plane can be found by considering equations (4) if $\eta \neq 0$ and equation (13) holds. When $I^2 - \gamma J = \gamma P^2 - 2IP + J = 0$, it is seen that $\Delta = \theta = 0$ and hence that the left-hand side of equation (13) reduces to $2430000\gamma^4\eta^4$ which does not vanish.

If $\eta = 0$, equations (9), (10), and (11) have a solution if and only if $\gamma P^2 - 2IP + J = 0$ also, and then $3D = -IP^2 + 2JP$. We have already encountered this case among the solutions of equations (3) and so no new cubic surface can be obtained from equations (4) when $\eta = 0$.

We may conveniently summarize our results in the following table, in which $\mathcal{D}$ is an abbreviation for the left-hand side of equation (13).

TABLE I

$I^2-\gamma J$	$\gamma P^2-2IP+J$	$\gamma_u\gamma_v-\gamma\gamma_{uv}$	$\mathcal{D}$	(3)	(4)
*	*	*	*	2	1
*	*	*	0	2	0
*	*	0	*	2	0
*	*	0	0	Impossible	
*	0	*	*	1	1
*	0	*	0	1	0
*	0	0	*	Impossible	
*	0	0	0	2	0
0	*	*	*	1	1
0	*	*	0	1	0
0	*	0	*	1	0
0	*	0	0	Impossible	
0	0	*	*	0	1
0	0	*	0	Impossible	
0	0	0	*	Impossible	
0	0	0	0	1	0

In the first four columns a $*$ is entered if the quantity at the head of the column is different from zero and a 0 is entered if it vanishes. In the fifth column is entered the number of cubic surface (2) with a double point not in the tangent plane which can be obtained by considering equations (3), and in the sixth column is entered the number of such additional cubic surfaces which can be obtained by considering equations (4). We have already seen that when $I^2-\gamma J=\gamma P^2-2IP+J=0$, then $\mathcal{D}$ and η either both vanish or neither vanishes. Hence the cases in the fourteenth and fifteenth lines do not occur. Moreover, when $\eta=0$ it is easy to see that $\mathcal{D}=-\Delta(9\theta^2-1600\gamma^4\Delta)^2$, so that $\mathcal{D}$ and $\gamma P^2-2IP+J$ either both vanish or neither vanishes. Hence the cases in the fourth, seventh, twelfth, and fifteenth lines do not occur either. In line eight one of the two surfaces is the surface S itself which is cubic and not Cayley's cubic scroll. In line sixteen the surface is S itself, which is Cayley's cubic scroll. In both cases there is an entire line of double points not in the tangent plane.

It is worth noticing that if none of the four quantities $I^2-\gamma J$, $\gamma P^2-2IP+J$, $\gamma_u\gamma_v-\gamma\gamma_{uv}$, and $\mathcal{D}$ vanishes, then there are three cubic surfaces (2) which have a double point not in the tangent plane. If exactly one of these quantities vanishes there are two such cubic surfaces. If exactly two vanish there is one such cubic surface. If exactly three vanish, there are two such cubic surfaces, one of which is S itself and is not Cayley's cubic scroll. If all four vanish, there is one such cubic surface, namely S itself, which is Cayley's cubic scroll.

Let us now study the geometry in the tangent plane of S. It has been shown [1] that the tangent plane intersects S in a curve with two branches, one of which is the generator $y=z=0$, and the other is

$$(14)\qquad \begin{aligned} x &= \frac{1}{3}\gamma y^2 + \frac{1}{12}\gamma_v y^3 + \frac{1}{180}(3\gamma_{vv} - 10\gamma\gamma_u)y^4 \\ &\quad - \frac{1}{360}(10\gamma_u\gamma_v + 5\gamma\gamma_{uv} - \gamma_{vvv})y^5 + \cdots, \quad z = 0, \end{aligned}$$

and that the equation of the osculating conic of this curve is

$$(15)\qquad z = Px^2 + x - \frac{1}{3}\gamma y^2 - \frac{1}{4}(\gamma_v/\gamma)xy = 0.$$

There exists a one-parameter family of noncomposite cubic curves with a node at the origin and asymptotic tangents for nodal tangents each of which has fifth order contact with the curve (14). The equation of such a cubic curve is

$$(16)\qquad z = kx^3 - \frac{1}{3}\gamma y^3 + Px^2y - \frac{1}{4}(\gamma_v/\gamma)xy^2 + xy = 0,$$

where k is the parameter. The points of inflexion of this nodal cubic curve lie on the line

$$(17)\qquad z = Px - \frac{1}{4}(\gamma_v/\gamma)y + 1 = 0,$$

some one of the inflexions lying on each of the lines $3kx^3=\gamma y^3$. The line (17) is tangent to the osculating conic (15) at the point whose homogeneous coordinates are $(-P, 1, 0, 0)$. This is the point characterized above as the only double point of a cubic surface (2) which lies in the tangent plane. It may also be characterized as the intersection different from the origin of the generator $y=z=0$ and the osculating conic (15).

Since the point $(-P, 1, 0, 0)$ is a double point of the cubic surface (2), the polar quadric of $(-P, 1, 0, 0)$ with respect to (2) must decompose into two planes. These planes are

$$Px_4 + x_3 = 0, \qquad Mx_4 - \frac{1}{4}(\gamma_v/\gamma)x_3 + Px_2 + x_1 = 0.$$

The first of these planes intersects the tangent plane in the generator, and the second intersects the tangent plane in the line (17).

The nodal cubic curve (16) has sixth order contact with the curve (14) if and only if $k=27\mathcal{C}/\gamma^3$, where $\mathcal{C}$ was defined in [1] as

$$\mathcal{C} = [18\gamma\gamma_v\gamma_{vv} - 15\gamma_v^3 - 4\gamma^2\gamma_{vvv} - 20\gamma^2(\gamma_u\gamma_v - \gamma\gamma_{uv})]/1440\gamma^2.$$

Therefore, there exists a noncomposite nodal cubic curve, nodal tangents being the asymptotic tangents, with sixth order contact with the curve (14) if and only if $\mathcal{C}\neq 0$. If $\mathcal{C}=0$, the only such cubic decomposes into the generator $y=z=0$ and the conic (15) which then hyperosculates the curve (14).

If we do not insist that the nodal tangents be the asymptotic tangents, we can prove that there exists a one-parameter family of cubic curves with a node at the origin and sixth order contact with the curve (14). The equation of such a cubic curve is

$$\begin{aligned} z = (EP + k)x^3 - \frac{1}{3}\gamma y^3 + \left[P - \frac{1}{4}E(\gamma_v/\gamma)\right]x^2y \\ - \left[\frac{1}{3}E\gamma + \frac{1}{4}(\gamma_v/\gamma)\right]xy^2 + Ex^2 + xy = 0, \end{aligned} \tag{18}$$

where $k=27\mathcal{C}/\gamma^3$ and E is the parameter. The nodal tangents are

$$x = 0, \qquad y = -Ex, \tag{19}$$

and the line of inflexions is

$$(4E^2\gamma^2 + 12P\gamma)x + (8E\gamma^2 - 3\gamma_v)y + 12\gamma = 0. \tag{20}$$

As E varies this line of inflexion envelops a conic which turns out to be precisely the osculating conic (15), which thus receives another geometrical characterization. Moreover, the nodal tangent $y+Ex=0$ intersects the line of inflexion (20) in the point whose homogeneous coordinates are $(4E^2\gamma^2-3E\gamma_v-12P\gamma,\ 12\gamma,\ -12E\gamma,\ 0)$, the locus of which as E varies is also the osculating conic (15).

Bibliography

1. Wilkins, J. Ernest, Jr., *The contact of a cubic surface with a ruled surface*, Amer. J. Math. vol. 67 (1945) pp. 71–82.

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