ON UNIVERSAL MAPPINGS AND FREE TOPOLOGICAL GROUPS

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It has been observed[1] that constructions so apparently different as Kronecker products, extension of the ring of operators of a module, field of quotients of an integral domain, free groups, free topological groups, completion of a uniform space, Čech compactification enter in the same frame. We intend in this paper to explain a rather general process of construction which may be applied to most of the examples quoted above.

This paper will proceed axiomatically. In fact the problem under question (problem of a "universal mapping") can be only stated after a certain number of axioms. When the method of construction has been explained we shall illustrate it by the classical example of the completion of a uniform space. For more examples the reader is referred to a forthcoming book of N. Bourbaki. The same method gives also necessary and sufficient conditions for many imbedding problems. Both topological and algebraic examples will be given. In the last part of the paper our method of construction will be applied to Markoff's theory of free topological groups.[2]

1. **Problems of universal mappings.** Given a set E it is possible to define on it certain kinds of structures, that is structure of ring, field, topological space.[3] We shall denote by S or T certain kinds of structures. A set with a structure T will be called a *T-set*: if T is the structure of group the T-sets are the groups. An isomorphism for the structure T will be called a T-isomorphism:

T-mappings. Induced structures. Given a kind of structure T it happens very often that, for every pair E_1E_2 of T-sets, there has been defined a family of mappings of E_1 into E_2 satisfying the following axioms:

A_1. *Every T-isomorphism is a T-mapping.*

A_2. *If $f:E_1\rightarrow E_2$ and $g:E_2\rightarrow E_3$ are T-mappings, then the composite*

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[1] Unpublished manuscripts of N. Bourbaki.

[2] Markoff, Bull. Acad. Sci. USSR. vol. 9 (1945) pp. 3–64.

[3] For precise definitions of the words "structure," "kind of structure," "isomorphism" see N. Bourbaki, *Theorie des ensembles* (*Résultats*), Part 10, Paris, Herrmann, 1939.

mapping $g \circ f: E_1 \to E_3$ *is a T-mapping.*

A_3. *A necessary and sufficient condition for a one-to-one mapping f of* E_1 *onto* E_2 *to be a T-isomorphism is that f and* f^{-1} *be T-mappings.*

EXAMPLE. If T is the structure of group the T-mappings are the homomorphisms; if T is the structure of topological space the T-mappings are the continuous ones.

Let now σ and σ' be two structures T defined on E and $E' \subset E$ respectively. We shall say that σ' is *induced* by σ when:

I_1. *The injection of* E' *into* E *is a T-mapping.*

I_2. *If* $f: F \to E$ *is a T-mapping and if* $f(F) \subset E'$, *then f considered as mapping of F into* E' *is a T-mapping.*

I_1, I_2 and A_3 show immediately the *uniqueness* of the induced structure.

If $E' \subset E$ is capable of an induced structure we shall say that E' is *T-closed*. We suppose that the following axioms hold:

S_1. *A subset of E composed of all the elements where a family of T-mappings takes the same value is T-closed.*

S_2. *Any intersection of T-closed sets is T-closed.*

S_2 is a consequence of S_1 if we allow mappings which are not everywhere defined: The notion of T-closed set is obviously transitive. S_2 allows us to define the *T-closure* $\bar{E}'$ of a subset $E' \subset E$ as the intersection of all T-closed subsets containing E'. We shall suppose:

S_3. Cardinal $(\bar{E}') \leq$ certain function of cardinal (E'), a function which depends only on the structure T.

In most cases the function "$2^{2^{\text{card}(E')}}$" will be sufficient.

Axioms for the cartesian products. In many important cases it is possible, given a family (E_α) of T-sets, to define on the cartesian product $\prod_\alpha E_\alpha$ a structure T which satisfies the following conditions:

P_1. *The projections (on the components) are T-mappings.*

P_2. *If the* $f_\alpha: E \to E_\alpha$ *are T-mappings, the product mapping* $f: E \to \prod_\alpha E_\alpha$ *(defined by* $f(x) = (f_\alpha(x))$*) is a T-mapping.*

As easy consequences: the projections on the partial products are T-mappings; these partial products are T-closed; the given structure of the "coordinate axis" is induced by the product structure. Applying A_3 we see also that the product structure is *unique*.

Remark. The example of the structure of topological space shows easily that the axioms A, S, P are independent.

Statement and solution of the problem of universal mappings. Given two kinds of structures S and T, suppose we have defined the T-mappings, and also mappings of S-sets into T-sets, called the *(S-T)-mappings*, denoted by greek letters, and satisfying:

$(S\text{-}T)_1$. *The composite mapping* $f \circ \phi$ *of an (S-T)-mapping* ϕ *and of* *a T-mapping f is an (S-T)-mapping.*

(S-T)$_2$. *The product mapping of a family of (S-T)-mappings is an (S-T)-mapping.*

The structure T is supposed to satisfy the axioms A, S and P.

The problem we have in view ("problem of universal mappings" or "problem U") is the following: given any S-set E to find a T-set F_0 and an $(S\text{-}T)$-mapping ϕ_0 of E into F such that:

(U$_1$). *Every (S-T)-mapping ϕ of E into any T-set F has the form $\phi = f \circ \phi_0$ where f is a T-mapping of F_0 into F.*

It is clear that, if such an F_0 exists, the T-closure of $\phi_0(E)$ in F_0 will also satisfy (U$_1$). Therefore (S$_1$) there will exist a pair (F_0, ϕ_0) such that:

(U$_2$). *Two T-mappings of F_0 into F which coincide on $\phi_0(E)$ are identical.*

We deduce immediately from A$_3$ that a pair (F_0, ϕ_0) satisfying (U$_1$) and (U$_2$) is *uniquely determined* up to isomorphisms.

We now come to the *construction* of a pair (F_0, ϕ_0) satisfying (U$_1$). Consider the set of all $(S\text{-}T)$-mappings of E into all T-sets whose cardinal does not exceed the one indicated in S$_3$ ($2^{2^{\text{card } E}}$ in most cases). Let $\{\phi_\alpha\}$ be this set, ϕ_α mapping E into F_α. Let $F_0 = \prod_\alpha F_\alpha$, and ϕ_0 be the product mapping $x \rightarrow (\phi_\alpha(x))$ of E into F_0. F_0 is a T-set (P), and ϕ_0 an (S-T)-mapping ($(S\text{-}T)_2$). Let ϕ be an $(S\text{-}T\text{-})$mapping of E into F, F' the T-closure of $\phi(E)$ in F, i the injection of F' in F. By S$_3$, I$_2$ and (S-T) the contraction ϕ' of ϕ, mapping of E into F, is among the ϕ_α, say ϕ_{α_0}. Let p_0 be the projection of $\prod_\alpha F_\alpha$ onto F_{α_0}. We may write $\phi = i \circ \phi' = i \circ \phi_{\alpha_0} = i \circ p_0 \circ \phi_0$. Since $i \circ p_0$ is a T-mapping of $\prod_\alpha F_\alpha$ into F, the pair (F_0, ϕ_0) satisfies (U$_1$). Q.E.D.

EXAMPLE. The preceding construction applies to all the examples quoted in the introduction except the field of quotients of an integral domain (a product of fields being not a field). For these examples the reader is referred to a forthcoming book of N. Bourbaki.

We shall give only the example of the completion of a uniform space.[4] S is the structure of separated uniform space, T the structure of complete separated uniform space. The T- and $(S\text{-}T)$-mappings are the uniformly continuous ones. All our axioms are verified. Therefore, given a separated uniform space E, the preceding construction provides with a complete space F_0 and a uniformly continuous mapping ϕ_0 of E into F such that every uniformly continuous mapping of E is "induced" by a uniformly continuous mapping of F_0.

We shall prove that, in this case, ϕ_0 is a uniform structural isomorphism of E onto $\phi_0(E)$. In fact consider the family $\mathfrak{E}$ of all uni-

[4] N. Bourbaki, *Topologie générale*, chap. 2, Paris, Hermann, 1940.

formly continuous distances (not necessarily satisfying the separation axiom "$d(x, y)=0$ implies $x=y$") on E. Since the real line is a complete space the function f defined by fixing one of the arguments in the distance d is among the f_α. Since the uniform structure of E may be defined by the family $\mathfrak{E}$,[5] we see immediately—denoting by $\{f_\beta\}$ the subset of $\{f_\alpha\}$ composed of all mappings deduced from distances, and by π the projection of $\prod_\alpha F_\alpha$ onto $\prod_\beta F_\beta$—that $\pi \circ \phi_0$ is an isomorphism. Hence ϕ_0 is one-to-one and $\phi_0^{-1}=(\pi \circ \phi_0)^{-1} \circ \pi$ is uniformly continuous. Hence ϕ_0 is an isomorphism.

By definition of the T-closure (smallest complete, that is, closed subspace), we see that $\phi_0(E)$ is dense in F. We have therefore proved the existence and the uniqueness of the completion of a uniform space (provided we have defined the real line without completion, for example by the cuts process). If E is not separated one verifies easily that $\phi_0(E)$ is the associated separated space.

2. **Imbedding problems.** It often happens that the structure T is "richer" than the structure S, that is, that there exists a canonical process for giving a structure S to a T-set. In an example where $\phi_0(E)$ is, in the S-set F_0, capable of an induced structure S, arises, with the problem (U), an "imbedding problem": may we consider E as a subset of a set F, subset whose structure S is induced by the structure S of F canonically deduced from a structure T? We shall suppose that a T-mapping is also an S-mapping for the deduced structures S. As a consequence (A_3 for S) the operations "induced structure" and "deduced structure" *commute.*

If $\phi_0(E)$ is capable of an induced structure S, and if ϕ_0 is an S-isomorphism, the imbedding problem is solved by (F_0, ϕ_0). Let conversely (F, ϕ) be a solution of the imbedding problem, ϕ being an $(S$-$T)$-mapping. We can write $\phi=f \circ \phi_0$, f being a T-mapping of F_0 into F. Since ϕ is an S-isomorphism, and since $f|\phi_0(E)$ and ϕ_0 are S-mappings, it follows from A_2 and A_3 (for S) that $f|\phi_0(E)$ and ϕ_0 are S-isomorphisms. Therefore (F_0, ϕ_0) gives also a solution of the imbedding problem.

We may therefore consider F as a set of equivalence classes in F_0, the sets $\{a\}$ $(a \in \phi_0(E))$ being equivalence classes. The fact that this identification is not always trivial (and that the imbedding problem does not admit a unique solution) is shown by the example of the compactifications of a uniform space (F_0 being the Čech compactification). Therefore: *A necessary and sufficient condition for the imbedding problem of E to be possible is that ϕ_0 be an S-isomorphism.*

[5] N. Bourbaki, *Topologie générale*, chap. 9, part 1, Theorem 1, 1948.

EXAMPLES. (1) *Characterization of uniformizable spaces.*[6] The structures S and T are the structures of topological space and of compact space respectively. The S-, T-, (S-T)-mappings are the continuous ones. Let $\mathfrak{T}$ be the given topology on E, $\mathfrak{T}'$ the one induced on $\phi_0(E)$ by the product space topology. $\mathfrak{T}'$ and $\mathfrak{T}_0 = \phi_0^{-1}(\mathfrak{T}')$ are uniformizable. In general $\mathfrak{T}$ is finer than $\mathfrak{T}_0$. A necessary and sufficient condition for $\mathfrak{T}$ to be uniformizable is that $\mathfrak{T} = \mathfrak{T}_0$. Then F_0 is the Čech compactification of E. Using the fact that every compact space may be imbedded in some, finite or transfinite, cube, one shows easily that the condition "$\mathfrak{T} = \mathfrak{T}_0$" is equivalent with the complete regularity of E.

(2) *Alexandroff's T_2-space.*[7] S is the structure of topological space, T the structure of T_2-space, the S-, T-, (S-T)-mappings are the continuous ones. Then $\phi_0(E)$ is the "greatest possible" identification space of E which is a T_2-space.

Remark. It may happen, in some cases where our axioms are not all fulfilled (in particular S_2 relative to intersections of T-closed sets), that one can however construct ϕ_0, while F_0 is "too big."

(1). Let S be the structure of ring, T the structure of semi-simple ring. The S-, T-, (S-T)-mappings are the homomorphisms *into.* In this case $\phi_0^{-1}(\{0\})$ is the *extension radical* $\mathfrak{R}$ of the ring E.[8] If $\mathfrak{R} = 0$, E may be imbedded in a semi-simple ring and conversely. The structure of $\mathfrak{R}$ may be obtained by studying a sufficient number of homomorphisms of E into semi-simple rings. Let $\mathfrak{a}$ be the two-sided ideal $T \cap \bigcap_p pE$ (T: ideal of the elements of finite order in the additive group of E; p: prime number). Every semi-simple ring being contained in a product of full matrix rings over s-fields, it is clear that $\phi(\mathfrak{a}) = 0$ for every homomorphism ϕ of E into a semi-simple ring. Hence $\mathfrak{a} \subset \mathfrak{R}$. On the other hand the rings E/pE and the Kronecker product $E/T \otimes Q$ of E/T by the rational field Q are algebras over the prime fields. But every algebra is a subalgebra of a full matrix algebra (adjoin a unit element, and consider the left regular representation). Hence $\mathfrak{R} \subset pE$, $\mathfrak{R} \subset T$. Therefore $\mathfrak{R} = \mathfrak{a}$.

(2) If we restrict the (S-T)-mappings to be the homomorphisms *onto* semi simple rings, it is easily seen that $\phi_0^{-1}(\{0\})$ is the *radical* of E.[9]

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[6] P. Samuel, *Ultrafilters and compactification of uniform spaces*, Princeton thesis, 1947.

[7] P. Alexandroff, *Bikompakte Erweiterungen von Räumen*, Rec. Math. (Mat. Sbornik) N.S. (1939).

[8] O. Goldman. *Semi-simple extensions of rings*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1028–1032.

[9] N. Jacobson, *Radical and semi simplicity for arbitrary rings*, Amer. J. Math. vol. 67 (1945) pp. 300–320.

3. Free topological groups. As a more elaborate example of our general method of construction, we shall give a sketchy treatment of Markoff's theory of free topological groups. Only very easy proofs have been omitted.

S is the structure of topological space, T the structure of separated topological group (abelian, or precompact, if desired). The (S-T)-mappings are the continuous ones, the T-mappings are the continuous homomorphisms. All the axioms A, S, P are satisfied. Applying the general construction to the topological space E, we get for F_0 a separated topological group (abelian, or precompact, if desired) denoted $G(E)$, ($GA(E)$, $GC(E)$). By construction $G(E)$ is such that:

(1) *For every continuous mapping ϕ of E into a topological group G, there exists a continuous homomorphism g of $G(E)$ into G such that $\phi = g \circ \phi_0$.*

(2) *Two continuous homomorphisms of $G(E)$ into a topological group H which coincide on $\phi_0(E)$ are identical; $\phi_0(E)$ generates $G(E)$; $G(E)$ is uniquely determined by E.*

Similar properties hold for $GA(E)$ and $GC(E)$. The uniqueness shows that $GA(E) \approx G(E)/C$, C being the closure of the commutator subgroup of $G(E)$. If E is discrete, $G(E)$ and $GA(E)$ are the classical free group and free abelian group generated by the set E.

THEOREM 1. *A necessary and sufficient condition for ϕ_0 to be a homeomorphism is that E be completely regular. The uniform structures induced on the $\phi_0(E)$ are the universal uniform structure (for $G(E)$ and $GA(E)$), and the Čech uniform structure (for $GC(E)$).*

The necessity is clear. The sufficiency comes easily from the fact that one has sufficiently many mappings continuous of E into the real line R or the unit circle T.

THEOREM 2. *If E' is a subspace of the completely regular space E, $G(E')$, $GA(E')$ and $GC(E')$ are algebraically isomorphic to subgroups of $G(E)$, $GA(E)$ and $GC(E)$, and have finer topologies.*

If we extend the injection $i: E' \to E \to G(E)$, we get a continuous homomorphism $g: G(E') \to G(E)$. g is shown to be one-to-one by proving that, given a word $\prod_i a_i^{n_i}$ ($a_i \in E'$), there exists a topological group G and a continuous mapping f of E' into G such that $\prod_i f(a_i)^{n_i} \neq I$. One may take for G the unimodular orthogonal group $O_3[R]$ which contains free groups with as many generators as desired, and which is arcwise connected. For $GA(E')$ an R^n is good enough.

As corollaries, we have:

(a) *$G(E)$ and $GC(E)$ (resp. $GA(E)$) have the algebraic structure of* *the free (resp. free abelian) group generated by the underlying set of E.*

(b) *The commutator subgroup C of $G(E)$ is closed; $GA(E) \approx G(E)/C$.*

(c) $\phi_0(E)$ *is a closed subset of $G(E)$ (resp. $GA(E)$, $GC(E)$).*

((c) is proved by imbedding E in a compact space.)

(d) *Every free group is algebraically isomorphic with a subgroup of a compact group.*

(e) *The topology of $G(E)$ is finer than the topology of $GC(E)$.*

When E is connected, the subgroup A of $G(E)$ (resp. $GA(E)$, $GC(E)$) composed of all the words of total degree zero is the connected component of the identity. If E is compact, A is a countable union of compact sets.

If $E' \subset E$, $G(E')$ is topologically isomorphic with a subgroup of $G(E)$ in the following cases (where continuous mappings of E' into sufficiently many groups can be extended to E):

(a) E' is a retract of E.

(b) E is the completion of E' for the universal uniform structure.

(c) E is normal and E' is a closed subset of E.

A counter-example is the following: E' is a discrete noncountable space, and E is its Čech compactification (consider the subgroup composed of the words of total degree zero).

The Markoff's schemes. By a *scheme* is meant a set $\mathfrak{S}$ of words formed from elements of E. We shall consider the continuous mappings $\{f_\gamma\}$ of E into topological groups such that $\prod_i f_\gamma(a_i)^{n_i} = 1$ for every word $\prod_i a_i^{n_i} \in \mathfrak{S}$. Such an f_γ is called a *realization of the scheme* $\mathfrak{S}$.

We apply the general construction to the mappings f_γ (with the usual restriction on cardinal numbers): we form the partial product $\prod_\gamma G_\gamma$ of $\prod_\alpha G_\alpha$, and construct the subgroup $G\mathfrak{S}(E)$ generated by the elements $(f_\gamma(x))$ $(x \in E)$. Let $\phi_{\mathfrak{S}}$ be the mapping $x \rightarrow (f_\gamma(x))$. It is clear that every realization of the scheme $\mathfrak{S}$ may be written $f = g \circ \phi_{\mathfrak{S}}$, where g is a continuous homomorphism of $G\mathfrak{S}(E)$. On the other hand such a pair $(G\mathfrak{S}(E), \phi_{\mathfrak{S}})$ is unique. If H denotes the closure of the invariant subgroup of $G(E)$ generated by $\mathfrak{S}$, $G\mathfrak{S}(E)$ is isomorphic with $G(E)/H$ (because of the uniqueness).

EXAMPLES. (a) $G(E)$ is defined by the empty scheme, $GA(E)$ by the set of commutators.

(b) If E has a structure of topological group, we may take for $\mathfrak{S}$ the "multiplication table" of E. A pair having the properties of $(G\mathfrak{S}(E), \phi_{\mathfrak{S}})$ being $(E$, identity), we deduce from the uniqueness that:

Every topological group E is isomorphic with a factor group of the free topological group generated by the underlying topological space of E.

(c) Taking for E the topological sum of two topological groups (with identity elements identified), and for $\mathfrak{S}$ the union of their multiplication tables, one gets the *free topological product* of the two groups.

The referee has pointed out to me that S. Kakutani (*Free topological groups and infinite direct products of topological groups*, Proc. Imp. Acad. Tokyo vol. 20 (1944) pp. 595–598) gives substantially the same proof of Markoff's theorem as I do.

Notice also that Nakayama's results (*A note on free topological groups*, Proc. Imp. Acad. Tokyo vol. 19 (1943) pp. 471–475) can be obtained by our method: his topology for $G(E)$, deduced from the continuous representations, is that of our $GC(E)$. If E is a uniform space, Nakayama's uniform free topological group is obtained by taking, as $(S\text{-}T)$-mappings, the uniformly continuous ones.

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