A NOTE ON THE DERIVATIVES OF INTEGRAL FUNCTIONS

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1. **Introduction.** Let $f(z)=\sum_0^\infty a_n z^n$ be an integral function of order ρ and lower order λ, and $M(r)=\max_{|z|=r}|f(z)|$; $M'(r)=\max_{|z|=r}|f'(z)|$. In a recent paper [1][1] I have proved the following two theorems.

THEOREM A. *If $f(z)$ be any integral function of order ρ then*[2]

$$\limsup_{r\to\infty} \frac{\log \{rM'(r)/M(r)\}}{\log r} = \rho. \tag{1.1}$$

THEOREM B. *If $f(z)=\sum a_n z^n$ be an integral function of lower order λ and $a_n \geqq 0$ then*

$$\liminf_{r\to\infty} \frac{\log \{rM'(r)/M(r)\}}{\log r} = \lambda.$$

The condition that the coefficients a_n be real and non-negative is unnecessary. The purpose of this note is to prove the following two theorems and to deduce a number of interesting results.

THEOREM 1. *If $f(z)$ be an integral function of lower order λ $(0\leqq\lambda\leqq\infty)$ then*

$$\liminf_{r\to\infty} \frac{\log \{rM'(r)/M(r)\}}{\log r} = \lambda. \tag{1.2}$$

THEOREM 2. *For any integral function $f(z)$ we have*

$$\begin{aligned}\liminf_{r\to\infty} M'(r)/M(r) &\leqq \liminf_{r\to\infty} \nu(r)/r \leqq \limsup_{r\to\infty} \nu(r)/r \\ &\leqq \limsup_{r\to\infty} M'(r)/M(r),\end{aligned} \tag{1.3}$$

$$\begin{aligned}\liminf_{r\to\infty} M^{(s+1)}(r)/M^{(s)}(r) &\leqq \liminf_{r\to\infty} \nu(r)/r \leqq \limsup_{r\to\infty} \nu(r)/r \\ &\leqq \limsup_{r\to\infty} M^{(s+1)}(r)/M^{(s)}(r) \qquad (s=1,2,3,\cdots),\end{aligned} \tag{1.4}$$

where $f^{(s)}(z)$ is the sth derivative of $f(z)$, $M^{(s)}(r)=\max_{|z|=r}|f^{(s)}(z)|$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] A glance at the proof [1, pp. 1–2] shows that the result (1.1) holds when ρ is infinite. An alternative proof of Theorem A is to employ Lemma 4 and relation (8) of my paper [1, p. 1].

and $\nu(r)$ *is the rank of the maximum term of* $f(z)$ *for* $|z|=r$.

2. Lemmas.

LEMMA 1. *Let*

(2.1) $F(x)$ *be a positive nondecreasing function for* $x > x_0$,

$$\liminf_{x\to\infty} F(x)/x = a \qquad (0 \leqq a < \infty); \tag{2.2}$$

then, corresponding to each pair of positive numbers b, c *satisfying the inequalities*

$$a < b; \qquad a/b < c < 1$$

there is a sequence $x_1, x_2, \cdots$, *tending to infinity such that*

$$F(x) < bx \qquad (cx_n \leqq x \leqq x_n).[3]$$

PROOF. For there is a sequence $x_1, x_2 \cdots$ such that

$$F(x_n) < bcx_n.$$

Hence if $cx_n \leqq x \leqq x_n$

$$F(x) \leqq F(x_n) < bcx_n \leqq bx.$$

LEMMA 2. *Let* (2.1) *hold,*

$$\limsup_{x\to\infty} F(x)/x = a_1 \qquad (0 < a_1 \leqq \infty); \tag{2.3}$$

then, corresponding to each pair of positive numbers b_1, c_1, *satisfying the inequalities*

$$b_1 < a_1; \qquad 1 < c_1 < a_1/b_1$$

there is a sequence $X_1, X_2, \cdots$ *tending to infinity such that*

$$F(x) > b_1 x \qquad (X_n \leqq x \leqq c_1 X_n).$$

For there is a sequence $\{X_n\}$ tending to infinity such that

$$F(X_n) > b_1 c_1 X_n.$$

If $X_n \leqq x \leqq c_1 X_n$

$$F(x) \geqq F(X_n) > b_1 c_1 X_n \geqq b_1 x.$$

LEMMA 3 [2, p. 22]. *Let* (2.1) *hold,*

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[3] It is always possible to choose the sequence $\{x_n\}$ such that two consecutive intervals have no common point. Similarly for the intervals of Lemmas 2, 3, and 4.

(2.4) $$\liminf_{x\to\infty} \log F(x)/\log x = \alpha \qquad (0 \leqq \alpha < \infty);$$

then, corresponding to each pair of positive numbers β, γ satisfying the inequalities

$$\alpha < \beta; \qquad \alpha/\beta < \gamma < 1$$

there is a sequence $y_1, y_2, \cdots$ tending to infinity such that

$$F(x) < x^{\beta} \qquad (y_n^{\gamma} \leqq x \leqq y_n).$$

LEMMA 4. *Let* (2.1) *hold,*

(2.5) $$\limsup_{x\to\infty} \log F(x)/\log x = \alpha_1 \qquad (0 < \alpha_1 \leqq \infty);$$

then, corresponding to each pair of positive numbers β_1, γ_1, satisfying the inequalities

$$\beta_1 < \alpha_1; \qquad 1 < \gamma_1 < \alpha_1/\beta_1$$

there is a sequence $Y_1, Y_2, \cdots$ tending to infinity such that[4]

$$F(x) > x^{\beta_1} \qquad (Y_n \leqq x \leqq Y_n^{\gamma_1}).$$

For there is a sequence $\{Y_n\}$ such that

$$\log F(Y_n) > \beta_1\gamma_1 \log Y_n.$$

If $Y_n \leqq x \leqq Y_n^{\gamma_1}$

$$\log F(x) \geqq \log F(Y_n) > \beta_1\gamma_1 \log Y_n = \beta_1 \log Y_n^{\gamma_1} \geqq \beta_1 \log x.$$

3. **Proof of Theorem 1.** Let

$$\theta(r) = \frac{\log \{rM'(r)/M(r)\}}{\log r}.$$

If λ be infinite, we have from the inequality [3]

(3.1) $$M'(r) > (M(r) \log M(r))/(r \log r); \qquad r > r_0 = r_0(f)$$

that[5]

$$\liminf_{r\to\infty} \theta(r) = \infty.$$

We therefore suppose that $0 \leqq \lambda < \infty$. From the inequality (3.1) it follows that

(3.2) $$\liminf_{r\to\infty} \theta(r) \geqq \lambda.$$

[4] We can deduce Lemmas 1 and 2 from Lemmas 3 and 4.

[5] r_0 and n_0 are not necessarily the same at each occurrence.

If $\nu(r)$ denotes the rank of the maximum term of $f(z)$, then we have [2, p. 21]

$$\liminf_{r\to\infty} \log \nu(r)/\log r = \lambda.$$

Hence by Lemma 3, corresponding to each pair of positive numbers β, γ satisfying the inequalities $\lambda<\beta$; $\lambda/\beta<\gamma<1$, there is a sequence $y_1, y_2, \cdots$ tending to infinity such that

$$\nu(r) < r^{\beta} \qquad (y_n^{\gamma} \leqq r \leqq y_n).$$

Let E_n denote the set of points r $(y_n^{\gamma}\leqq r\leqq y_n)$ and $E=E_1+E_2+\cdots$. Let F denote the set of points r which lie [4, p. 105] outside a set of exceptional segments in which, for $r>R$, the variation of $\log r$ is less than $K\nu(R/k)^{-1/12}$. Since the variation of $\log r$ in E_n is

$$\log y_n - \gamma \log y_n = (1-\gamma) \log y_n,$$

which tends to infinity with n, there are points in E_n which do not belong to the set of exceptional segments. The set EF therefore contains a sequence $e_1, e_2, \cdots$ tending to infinity. At these points [4, p. 105] e_n

$$rM'(r) \sim M(r)\nu(r), \qquad r = e_n, \tag{3.3}$$

$$rM^{(s+1)}(r) \sim M^{(s)}(r)\nu(r). \tag{3.4}$$

Hence[5] for $n>n_0$

$$e_n M'(e_n)/M(e_n) < 2\nu(e_n) < 2e_n^{\beta}$$

and so $\liminf_{r\to\infty} \theta(r)\leqq\beta$ and since $\beta-\lambda$ can be chosen arbitrarily small we have

$$\liminf_{r\to\infty} \theta(r) \leqq \lambda$$

and so $\liminf_{r\to\infty} \theta(r)=\lambda$.

4. **Proof of Theorem 2.** Let $\liminf_{r\to\infty}\nu(r)/r=a$ and suppose first that $a<\infty$. Then if

$$a < b, \qquad a/b < c < 1,$$

we have, by Lemma 1, $\nu(r)<br$ $(cx_n\leqq r\leqq x_n)$. Let E_n denote the set of points r $(cx_n\leqq r\leqq x_n)$ and $E=E_1+E_2+\cdots$. The variation of $\log r$ in E_n is $\log x_n-\log cx_n=\log 1/c$ which is not less than $K\nu(R/k)^{-1/12}$ if R be large enough. The total variation of $\log r$ in the intervals $\sum_{p=1}^{n}E_p$ tends to infinity with n. Hence the set EF contains [4, p. 105]

a sequence $e_1', e_2', \cdots$ tending to infinity. For $r=e_n'$ $(n>n_0)$

$$M'(r)/M(r) \sim \nu(r)/r < b, \tag{4.1}$$

$$M^{(s+1)}(r)/M^{(s)}(r) \sim \nu(r)/r < b \qquad (s = 1, 2, 3, \cdots). \tag{4.2}$$

Hence

$$\liminf_{r\to\infty} M'(r)/M(r) \leqq b.$$

Since $b-a$ can be chosen arbitrarily small

$$\liminf_{r\to\infty} M'(r)/M(r) \leqq a$$

which certainly holds if $a=\infty$. Also

$$\liminf_{r\to\infty} M^{(s+1)}(r)/M^{(s)}(r) \leqq a \qquad (s = 1, 2, \cdots).$$

Let $\limsup_{r\to\infty} \nu(r)/r=a_1$ and suppose $a_1>0$. Let $b_1<a_1$, $1<c_1<a_1/b_1$. If G denotes the set of points formed by the intervals of Lemma 2 the set GF contains a sequence $g_1, g_2, \cdots, g_n, \cdots$ tending to infinity. For $r=g_n$ $(n>n_0)$

$$M'(r)/M(r) \sim \nu(r)/r > b_1, \tag{4.3}$$

$$M^{(s+1)}(r)/M^{(s)}(r) \sim \nu(r)/r > b_1 \qquad (s = 1, 2, \cdots). \tag{4.4}$$

Hence $\limsup_{r\to\infty} M'(r)/M(r) \geqq a_1$,

$$\limsup_{r\to\infty} M^{(s+1)}(r)/M^{(s)}(r) \geqq a_1$$

which hold if $a_1=0$. Hence the theorem follows.

5. **Applications.** We have from (1.1) and (1.2)

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log \nu(r)}{\log r} &= \rho = 1 + \limsup_{r\to\infty}\frac{\log \{M'(r)/M(r)\}}{\log r}\\ &= 1 + \limsup_{r\to\infty}\frac{\log \{M^{(s+1)}(r)/M^{(s)}(r)\}}{\log r},\end{aligned} \tag{5.1}$$

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log \nu(r)}{\log r} &= \lambda = 1 + \liminf_{r\to\infty}\frac{\log \{M'(r)/M(r)\}}{\log r}\\ &= 1 + \liminf_{r\to\infty}\frac{\log \{M^{(s+1)}(r)/M^{(s)}(r)\}}{\log r}\end{aligned} \tag{5.2}$$

$$(s = 1, 2, \cdots).$$

Let s denote any fixed positive integer and C, C_1 two positive constants.

(5.3) *If* $M'(r) \geqq CM(r)$ *for all* $r > r_0$ *then either* $\lambda > 1$ *or* $\lambda = 1$ *and* $\liminf_{r\to\infty} \nu(r)/r \geqq C$.

From (1.3) we have $\liminf_{r\to\infty} \nu(r)/r \geqq C$. Hence $\lambda \geqq 1$.

REMARK (i). This is a best possible result in the sense that there are functions for which $M'(r) \geqq CM(r)$ and $\nu(r) \sim Cr$. We may take for instance $f(z) = \exp(Cz)$.

(ii) The converse—if $\liminf_{r\to\infty} \nu(r)/r \geqq C$ then $M'(r) \geqq CM(r)$ for all $r > r_0$—is false. For consider $f(z) = \cosh Cz$. Here $\lambda = 1 = \rho$, $\nu(r) \sim Cr$.

$$\frac{M'(r)}{M(r)} = C\frac{e^{Cr} - e^{-Cr}}{e^{Cr} + e^{-Cr}} < C \qquad \text{for all } r > 0,$$

$$\frac{M''(r)}{M'(r)} = C\frac{e^{Cr} + e^{-Cr}}{e^{Cr} - e^{-Cr}} > C \qquad \text{for all } r > 0,$$

and so on.

(5.4) *If* $M^{(s+1)}(r) \geqq CM^{(s)}(r)$, $r > r_0$, *then* $\lambda \geqq 1$ *and* $\liminf_{r\to\infty} \nu(r)/r \geqq C$.

(5.5) *If* $M'(r) \leqq C_1M(r)$, *or if* $M^{(s+1)}(r) \leqq C_1M^{(s)}(r)$, *for all* $r > r_0$, *then either* $\rho < 1$ *or* $\rho = 1$ *and* $\limsup_{r\to\infty} \nu(r)/r \leqq C_1$. This follows from Theorem 2. If $f(z)$ is of order 1 then [5, p. 81] it follows that $\limsup_{r\to\infty} \log M(r)/r \leqq C_1$.

Let $\phi(r)$ be any function, nondecreasing and positive for $r > r_0$, and such that $\log \phi(r)/\log r$ tends to zero as r tends to infinity.

(5.6) *If* $M'(r) \geqq (1/\phi(r))M(r)$, *or if* $M^{(s+1)}(r) \geqq (1/\phi(r))M^{(s)}(r)$ *for a sequence of values of* r *tending to infinity, then* $\rho \geqq 1$.

This follows from (5.1). If this hypothesis holds for all $r > r_0$ then from (5.2) we get $\lambda \geqq 1$.

(5.7) *If* $M'(r) \leqq \phi(r)M(r)$, *or if* $M^{(s+1)}(r) \leqq \phi(r)M^{(s)}(r)$, *for a sequence of values of* r *tending to infinity, then* $\lambda \leqq 1$.

This follows from (5.2). If this hypothesis holds for all $r > r_0$ then from (5.1) we get $\rho \leqq 1$.

(5.8) *If* $1/\phi(r) \leqq M'(r)/M(r) \leqq \phi(r)$ *or if* $1/\phi(r) \leqq M^{(s+1)}(r)/M^{(s)}(r) \leqq \phi(r)$ *for all* $r > r_0$ *then* $\lambda = \rho = 1$.

This follows from (5.6) and (5.7).

(5.9) *If* $\rho < 1$ *then*

$$M(r) > \phi(r)M'(r) > \phi^2(r)M''(r) > \cdots > \{\phi(r)\}^s M^{(s)}(r)$$

for all $r > r_0$.

This follows[6] from (5.1).

[6] Cf. (5.6) above.

(5.10) *If* $\lambda > 1$ *then*

$$M(r) < \frac{1}{\phi(r)} M'(r) < \frac{1}{\phi^2(r)} M''(r) < \cdots < \frac{1}{\{\phi(r)\}^s} M^{(s)}(r)$$

for all $r > r_0$

This follows[7] from (5.2).

(5.11) *If* $\lambda = 1$ *and* $\liminf_{r\to\infty} \nu(r)/(r \log r) > 1$ *then*[8] $M(r) < M'(r) < M''(r) < \cdots < M^{(s)}(r)$ *for all* $r > r_0$. If $\liminf_{r\to\infty} \nu(r)/(r \log r) = l > 1$ then since

$$\log M(r) > \int_{r_0}^{r} \{\nu(t)/t\} dt,$$

$$\liminf_{r\to\infty} \log M(r)/(r \log r) > 1.$$

Hence for all $r > r_0$

$$M'(r)/M(r) > \log M(r)/(r \log r) > 1,$$

$$M''(r)/M'(r) > \log M'(r)/(r \log r) > \log M(r)/(r \log r) > 1,$$

and so on.

(5.12) *If* $\lambda = 1$ *and* $\liminf_{r\to\infty} \nu(r)/r < 1$ *there is a sequence of numbers r tending to infinity for which*

$$M(r) > M'(r) > \cdots > M^{(s)}(r).$$

Let $\liminf_{r\to\infty} \nu(r)/r = a$ and $a < b < 1$. The result follows from (4.1) and (4.2). This result does not hold if $\liminf_{r\to\infty} \nu(r)/r \geqq 1$. In fact for the function $f(z) = \cosh z$, $\nu(r) \sim r$ and the sequence $\{M^{(n)}(r)\}$ $(n = 0, 1, 2, \cdots)$ is not monotonic for any $r > 0$.

(5.13) *If* $\lambda < 1$, *there is a sequence of numbers r tending to infinity for which*

$$M(r) > \phi(r)M'(r) > \phi^2(r)M''(r) > \cdots > \{\phi(r)\}^s M^{(s)}(r).$$

This follows from (3.3) and (3.4).

(5.14) *If* $\rho = 1$ *and* $\limsup_{r\to\infty} \nu(r)/r > 1$, *there is a sequence of numbers r tending to infinity for which*

$$M(r) < M'(r) < M''(r) < \cdots < M^{(s)}(r).$$

This follows from (4.3) and (4.4). It does not hold if $\limsup_{r\to\infty} \nu(r)/r \leqq 1$.

(5.15) *If* $\rho > 1$, *there is a sequence of numbers r tending to infinity*

[7] Cf. (5.7) above.

[8] S. K. Bose has proved (5.11) with the hypothesis $\lambda > 2$.

for which

$$M(r) < \frac{1}{\phi(r)} M'(r) < \frac{1}{\phi^2(r)} M''(r) < \cdots < \frac{1}{\{\phi(r)\}^s} M^{(s)}(r).$$

Let $1 < \beta_1 < \rho$; $1 < \gamma_1 < \rho/\beta_1$; H the set of intervals of Lemma 4. The set FH contains a sequence of numbers r tending to infinity for which the above inequality holds.

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