RATIONAL HARMONIC CURVES

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1. **Introduction.** We shall study curves related to rational fractional functions of a complex variable. This will generalize known results of curves related to rational integral functions.

Curves defined by setting the real part of a polynomial (rational integral function) in the complex variable $u=x+iy$ equal to zero are well known. These had been studied initially by Briot and Bouquet, and Bôcher; and finally characteristic properties have been given by Kasner. These curves have been called *algebraic potential curves* by Kasner, and this term is employed in later papers by Loria and in the German *Encyclopedia*. But we shall find it more convenient to use the term: *polynomial harmonic or polynomial potential curves.*

We define a *rational harmonic or rational potential curve* to be the locus obtained by setting the real part of a rational fractional function of a complex variable $u=x+iy$ equal to zero. The class of rational harmonic curves of course includes the class of polynomial harmonic curves.

We shall obtain various geometric properties of rational harmonic curves. These generalize corresponding results of Briot and Bouquet, and Kasner concerning the polynomial potential curves. We shall prove that *the real asymptotes of a rational potential curve are concurrent and make equal angles with one another*; *the remaining asymptotes are minimal.* This condition is only necessary but not sufficient. We do find a characteristic property of rational potential curves by studying the related focal properties. In the final part of our paper, we study the Schwarzian reflection with respect to a rational harmonic curve. *The satellite of a rational harmonic curve is itself.* This result gives the largest known class of self-satellite algebraic curves.

2. **Theorems of Briot and Bouquet, and Kasner concerning polynomial potential curves.** For purposes of contrast, certain theorems concerning polynomial harmonic curves will be stated.

The theorem of Briot and Bouquet concerning the asymptotes of a polynomial harmonic curve is as follows.[1]

The n asymptotes of a polynomial potential curve of degree n are all

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[1] Briot and Bouquet, *Theorie des fonctions elliptiques*, vol. 4, Paris, Gauthier-Villar, 1875, chap. 2, p. 226. See also Bôcher, Göttingen prize memoir.

real, concurrent, and disposed symmetrically about their common point, the angle between consecutive asympotes being π/n.

In all cases, the point of concurrence O of the asymptotes is a *center* of the curve. That is, if any line is drawn through O, the sum of the distances measured from O on one side of O is the same as the corresponding sum for the points on the other side. This follows from the fact that when the origin of coordinates is taken at O, all the terms of degree $(n-1)$ disappear.

Kasner noted that this result is not characteristic for polynomial harmonic curves of degree $n \geqq 4$. Some of Kasner's characterizations of polynomial harmonic curves are here stated.[2]

An algebraic curve is polynomial potential if and only if it is apolar to the fundamental conic of euclidean geometry which consists of the circular points I and J at infinity.

The polar curves of a circular point I (or J) with respect to a polynomial potential curve degenerate into sets of straight lines passing through the other circular point J (or I), and conversely.

An algebraic curve is polynomial harmonic when and only when the polar conics with respect to the curve are rectangular hyperbolas.

All the polars of polynomial potential curves are also polynomial potential curves.

Any curve of degree n is polynomial potential if and only if it passes through the n^2 *foci of a curve of class n.*

Each polynomial harmonic curve of degree n passes through the foci of an infinite number of systems of confocal curves of class n.

Two algebraic curves of degree n are conjugate polynomial potential curves if and only if they intersect orthogonally in the foci of a curve of class n.

Kasner has given also various geometric characterizations of polynomial harmonic surfaces in space.[3]

3. **Our theorem concerning the asymptotes of a rational potential curve.** Consider a rational conformal transformation of the form

$$
(1)\qquad
\begin{aligned}
U &= \frac{f(u)}{g(u)} = \frac{a_0u^r + a_1u^{r-1} + \cdots + a_r}{b_0u^s + b_1u^{s-1} + \cdots + b_s}, \\
V &= \frac{F(v)}{G(v)} = \frac{A_0v^r + A_1v^{r-1} + \cdots + A_r}{B_0v^s + B_1v^{s-1} + \cdots + B_s}
\end{aligned}
$$

[2] Kasner, *On the algebraic potential curves*, Bull. Amer. Math. Soc. vol. 7 (1901) pp. 392–399.

[3] Kasner, *Some properties of potential surfaces*, Bull. Amer. Math. Soc. vol. 8 (1902) pp. 243–248.

in the conjugate complex variables $u=x+iy$ and $v=x-iy$. Of course, A_j is the conjugate of a_j for $j=0, 1, 2, \cdots, r$, and B_k is the conjugate of b_k for $k=0, 1, 2, \cdots, s$. The leading coefficients (a_0, b_0, A_0, B_0) are assumed to be all different from zero. The polynomials $f(u)$ and $g(u)$ (and also $F(v)$ and $G(v)$) have no common factors.

We define a *rational harmonic curve* as the locus obtained by setting the real part of the rational fractional function $U=f(u)/g(u)$ equal to zero. Thus a rational harmonic curve is the image in the (x, y)-plane of the Y-axis defined in the (X, Y)-plane by $X=0$ or $U+V=0$ under the rational conformal transformation (1). Therefore *a rational harmonic or potential curve of total degree not exceeding $n=r+s$ is defined by the equation*

$$\begin{aligned}(2)\qquad &(a_0u^r + a_1u^{r-1} + \cdots + a_r)(B_0v^s + B_1v^{s-1} + \cdots + B_s)\\ &+ (A_0v^r + A_1v^{r-1} + \cdots + A_r)(b_0u^s + b_1u^{s-1} + \cdots + b_s) = 0,\end{aligned}$$

where we may assume that the integers r and s are such that $r \geqq s \geqq 0$ so that the degree of $U=f(u)/g(u)$ (or $V=F(v)/G(v)$) is $r \geqq 1$. For otherwise, (1) followed by an inversion and reflection will define the same rational harmonic curve (2). (2) may be written in the form

$$(3)\qquad f(u)G(v) + F(v)g(u) = 0.$$

Of course, the left-hand member of equation (2) or (3) is a polynomial in (u, v) or (x, y), of total degree which is not greater than $n=r+s$. *This polynomial does not satisfy the Laplace equation; but we have proved elsewhere that it does obey a certain partial differential equation of fourth order.*[4]

In particular, we can show by (2) that *the only real rational harmonic conic sections are the circles and the rectangular hyperbolas.* Of course, the latter are the only real polynomial harmonic curves of second degree.

Now we shall state the following result. The proof is given in §4.

FUNDAMENTAL THEOREM. *Let a rational potential curve be defined by setting equal to zero the real part of a rational fractional function of a complex variable $u=x+iy$ of degree $r \geqq 1$. This is given by an equation of the form $P(x, y)=0$, where $P(x, y)$, in general not harmonic, is a polynomial of degree $n=2r-k$ where $0 \leqq k \leqq r$. There are k real asymptotes all of which pass through a common point and make equal angles*

[4] Kasner and DeCicco, *A partial differential equation of fourth order connected with rational functions of a complex variable,* Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 326–328. See also a forthcoming paper, *Partial differential equations related to rational functions of a complex variable.* Duke Math. J. (1947).

with one another. The angle between consecutive asymptotes is π/k. *The remaining* $2(r-k)$ *asymptotes are minimal.*

Of course, if $k=0$ there are no real asymptotes, and all the asymptotes are minimal. On the other hand, if $k=1$ there is only one real asymptote.

When $k=r$, this result reduces to the theorem of Briot and Bouquet concerning the asymptotes of polynomial potential curves. Thus our fundamental theorem is an extension to rational fractional functions of the theorem of Briot and Bouquet.

4. **Proof of the fundamental theorem.** It is noted that if $r>s$, the degree of the rational potential curve (2) is exactly $n=r+s$. In this case, let $k=r-s$, so that the degree is $n=2r-k$. On the other hand, if $r=s$ the degree of the curve is equal to or less than $r+s=2r$ according as the expression $(a_0B_0+A_0b_0)$ is not or is zero.

If $r>s>0$, so that $n=r+s=2r-k$, the rational potential curve (2) passes through each of the circular points I and J at infinity. In general, there are $s=r-k$ branches of the curve which pass through each of these circular points. The curve intersects the line at infinity $s=r-k$ times in the circular point I, $s=r-k$ times in the circular point J, and in $n-2s=r-s=k$ other real points.

If $r=s>0$, and $a_0B_0+A_0b_0\neq 0$, the degree of the curve is $n=2r$ so that $k=0$. The curve intersects the line at infinity r times in the circular point I and r times in the circular point J.

Let $r=s>0$ and let $a_0B_0+A_0b_0=0$. There must exist a least integer k such that $0<k\leqq r$ for which the expression $a_0B_k+A_kb_0$, which is the coefficient of u^rv^{r-k} in (2), is different from zero. For otherwise, this would mean that (1) is degenerate. Since the conditions

$$\begin{gathered} a_0B_j + A_jb_0 = 0,\ A_0b_j + a_jB_0 = 0,\ \text{for } j = 0, 1, 2, \cdots, k-1; \\ a_0B_k + A_kb_0 \neq 0, \qquad A_0b_k + a_kB_0 \neq 0 \end{gathered} \tag{4}$$

are satisfied, we find that the equation (2) may be written in the form

$$\begin{aligned} &(a_0u^r + a_1u^{r-1} + \cdots + a_r)(B_kv^{r-k} + B_{k+1}v^{r-k-1} + \cdots + B_r) \\ &\quad + (A_0v^r + A_1v^{r-1} + \cdots + A_r) \\ &\qquad \cdot (b_ku^{r-k} + b_{k+1}u^{r-k-1} + \cdots + b_r) \\ &\quad + (B_0v^r + B_1v^{r-1} + \cdots + B_{k-1}v^{r-k+1}) \\ &\qquad \cdot (a_ku^{r-k} + a_{k+1}u^{r-k-1} + \cdots + a_r) \\ &\quad + (b_0u^r + b_1u^{r-1} + \cdots + b_{k-1}u^{r-k+1}) \\ &\qquad \cdot (A_kv^{r-k} + A_{k+1}v^{r-k-1} + \cdots + A_r) = 0. \end{aligned} \tag{5}$$

From (4) and (5), we find that the degree of the curve (5) is exactly $n=2r-k$. The curve intersects the line at infinity $(r-k)$ times in the circular point I, $(r-k)$ times in the circular point J, and in k other real points.

Thus unless our rational harmonic curve reduces to a polynomial harmonic curve, it is found that the rational harmonic curves always pass through the circular points I and J.

It can be proved by the equation (2) of a rational potential curve that if $r>s\geqq 1$ so that $n=r+s=2r-k$, the curve has $s=r-k$ asymptotes through the circular point I and $s=r-k$ asymptotes through the circular point J. Similarly the curve (5) where $r=s\geqq 1$ and the degree is $n=2r-k$, $0\leqq k\leqq r$, has $r-k$ asymptotes passing through I and $r-k$ asymptotes passing through J.

Next we consider the real asymptotes of the rational potential curve (2) where $r>s\geqq 0$ so that its degree is $n=r+s=2r-k$. The equation of any non-minimal line may be written in the form $v=mu+p$, where $m\neq 0$ and the inclination to the x-axis is $\theta=-(1/2i)\log m$. If this line is to be an asymptote of the curve (2), we find, upon substituting this into equation (2) and setting the coefficients of u^{r+s} and u^{r+s-1} equal to zero, the following conditions on m and p.

$$\begin{aligned} &a_0B_0m^s + A_0b_0m^r = 0,\\ (6)\qquad &a_1B_0m^s + a_0m^{s-1}(sB_0p + B_1)\\ &\qquad\qquad + A_0b_1m^r + b_0m^{r-1}(rA_0p + A_1) = 0. \end{aligned}$$

These equations are found to be equivalent to the system

$$\begin{aligned} &a_0B_0m^s + A_0b_0m^r = 0,\\ (7)\quad &A_0B_0(a_0b_1 - a_1b_0)m + a_0b_0A_0B_0(r - s)p\\ &\qquad\qquad + a_0b_0(A_1B_0 - A_0B_1) = 0. \end{aligned}$$

Thus if $r>s\geqq 0$ so that the degree of the curve is $n=r+s=2r-k$, there are $r-s=k$ real asymptotes, all of which pass through a common point and make equal angles with one another. The angle between consecutive asymptotes is π/k.

It is noted that by a translation, any rational harmonic curve is changed into a rational harmonic curve. Upon translating (2) so that the new origin O is the common point of intersection of the real asymptotes, it is found that the rational potential curve (2) assumes a similar form with the conditions

$$(8)\qquad a_0b_1 - a_1b_0 = 0, \qquad A_1B_0 - A_0B_1 = 0.$$

It follows by these conditions and (2) that *the point O is not in general a center of the curve.* The origin O is a center if and only if

$$\text{(9)} \qquad \text{either } a_0B_0 + A_0b_0 = 0 \quad \text{or} \quad a_1 = A_1 = b_1 = B_1 = 0.$$

For a polynomial potential curve, the latter conditions of (9) are valid, and thus the point O is a center.

If $r = s$, there are no non-minimal asymptotes when $a_0B_0 + A_0b_0 \neq 0$. Let k be the integer where $0 < k \leqq r$, for which the conditions (4) hold. The equation of the curve is given by (5). Its degree is $n = 2r - k$. Upon eliminating v between the equation of the line: $v = mu + p$, where $m \neq 0$, and (5), it is found that if this line is to be an asymptote of (5) the quantities m and p must satisfy the two conditions

$$\text{(10)} \qquad \begin{aligned} &(a_0B_k + A_kb_0)m^{r-k} + (A_0b_k + a_kB_0)m^r = 0, \\ &m^{r-k-1}[(r - k)(a_0B_k + A_kb_0)p + (a_0B_{k+1} + A_{k+1}b_0)] \\ &\qquad + m^{r-k}[a_1B_k + A_kb_1] + m^r[A_0b_{k+1} + a_{k+1}B_0] \\ &\qquad + m^{r-1}[r(A_0b_k + a_kB_0)p + (A_1b_k + a_kB_1)] = 0. \end{aligned}$$

These equations are equivalent to the system

$$\text{(11)} \qquad \begin{aligned} &(a_0B_k + A_kb_0) + (A_0b_k + a_kB_0)m^k = 0, \\ &k(A_0b_k + a_kB_0)(a_0B_k + A_kb_0)p \\ &\qquad + [(a_0B_k + A_kb_0)(A_0b_{k+1} + a_{k+1}B_0) \\ &\qquad\qquad - (A_0b_k + a_kB_0)(a_1B_k + A_kb_1)]m \\ &\qquad + [(a_0B_k + A_kb_0)(A_1b_k + a_kB_1) \\ &\qquad\qquad - (A_0b_k + a_kB_0)(a_0B_{k+1} + A_{k+1}b_0)] = 0. \end{aligned}$$

Thus for the rational potential curve (5) *of degree $n = 2r - k$, there are k real asymptotes, all of which pass through a common point and make equal angles with one another. The angle between consecutive asymptotes is π/k.*

It is found by (11) that the common center of the real asymptotes is not in general a center of the curve (5).

This completes the proof of our fundamental theorem.

5. **Other properties of rational potential curves.** An examination of equation (3) which defines a rational harmonic curve yields:

An algebraic curve of degree n not exceeding $(r+s)$, where $r > 1$ and $s > 1$, is rational harmonic if and only if it passes through the r^2 foci of a curve C_r of class r and the s^2 foci of a curve C_s of class s such that no minimal line contains a focus of C_r and a focus of C_s.

Of course, each rational harmonic curve of degree $n \leqq r+s$ passes through the r^2 foci of an infinite number of confocal curves C_r of class r and the s^2 foci of an infinite number of confocal curves C_s of class s.

If the real and imaginary parts of an analytic function of a complex variable are set equal to zero, the resulting curves may be called conjugate harmonic. Thus a pair of algebraic curves are *conjugate rational harmonic* if their equations are of the form

$$(12)\qquad f(u)G(v) + F(v)g(u) = 0, \qquad f(u)G(v) - F(v)g(u) = 0,$$

where the various functions appearing in these equations are the polynomials defined by equations (1).

A pair of algebraic curves of degree $n \leqq r+s$ are conjugate rational harmonic if and only if they intersect orthogonally in the r^2 foci of a system of confocal curves C_r of class r and in the s^2 foci of another system of confocal curves C_s of class s, such that no minimal line contains a focus of C_r and a focus of C_s.

It is remarked that the polar curves of a rational harmonic curve are not in general rational harmonic.[5]

6. **Satellite theory.** A Schwarzian reflection or conformal symmetry may be defined as a reverse conformal transformation of period two. It results that a Schwarzian reflection leaves fixed the points of a unique analytic curve. If the curve is given in minimal coordinates by the equation

$$(13)\qquad \phi(u, v) = 0,$$

the Schwarzian reflection T with respect to this curve is obtained by solving for (U, V) the equations

$$(14)\qquad \phi(U, v) = 0, \qquad \phi(u, V) = 0.$$

Kasner developed the geometry in the large of the Schwarzian reflection T with respect to a general algebraic curve; that is, when $\phi(u, v)$ is a general polynomial of degree n. *In general, the degree of the Schwarzian reflection T is n^2.* In general, the image of the algebraic curve (13) under the Schwarzian reflection T as given by the equation (14) is not only the curve itself but also a new algebraic curve which Kasner termed the *satellite* of the given curve (13). In particular, the satellite of a conic is a confocal conic.[6]

[5] Kasner and DeCicco, *Rational functions of a complex variable and related potential curves*, Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 280–282.

[6] Kasner, *La satelite conforme de una curva algebraica general*, Revista de la Unión Matemática Argentina vol. 2 (1946) pp. 77–83. Also *Algebraic curves, symmetries, and satellities*, Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 250–252.

The Schwarzian reflection T with respect to the rational harmonic curve (3) is

$$(15) \qquad f(U)G(v) + F(v)g(U) = 0, \quad f(u)G(V) + F(V)g(u) = 0.$$

The Schwarzian reflection T with respect to a rational harmonic curve of degree $n = 2r - k$, where $0 \leqq k \leqq r$, is of degree r^2.

In order to find the satellite of the rational potential curve (3), we have to eliminate (u, v) from the equations (3) and (15). It is found that the result yields the equation (3) where u is replaced by U and v is replaced by V.

The satellite of a rational potential curve is the original curve itself.

These new results are exact extensions of the corresponding theorems developed by Kasner concerning polynomial potential curves, to our rational potential curves.

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