ABSOLUTE AND UNCONDITIONAL CONVERGENCE

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A series $\sum_{n=1}^{\infty} x_n$ whose terms are elements of a Banach space B is said to converge *absolutely* if $\sum |x_n|$ converges, *unconditionally* if every rearrangement converges. If a series converges absolutely it converges unconditionally; absolute convergence and unconditional convergence are equivalent in any finite-dimensional Euclidean space, whereas in L^2, for instance, there are simple examples of series which converge unconditionally but not absolutely. The question in what spaces the two definitions are equivalent is left open by Banach [**1**, p. 240].[1] It may be conjectured that unconditional convergence does not imply absolute convergence in any infinite-dimensional space (cf. [**2**, p. 30]), but the problem does not seem to have been explicitly treated. The purpose of this note is to obtain a criterion for the equivalence of the two notions in a given Banach space. As examples of its use we shall show that unconditional convergence does not imply absolute convergence in the spaces L and l. The result for L is already known (see for example [**3**, p. 45]), but the result for l has not, to the author's knowledge, been stated elsewhere.

We shall need the following definitions. Let S be any (finite) sequence of elements $\xi_1, \xi_2, \cdots, \xi_p$ in B. We do not require that the ξ_i be all distinct, and we shall understand by *addition*, S_1+S_2, the mere adjoining of the two sequences. Thus we might write $(1, 2, 2, 3)+(3, 2, 1)=(1, 2, 2, 3, 3, 2, 1)$. By cS (c real) we understand the sequence $c\xi_1, c\xi_2, \cdots, c\xi_p$. We shall use two norms,

$$|S| = \sum |\xi_i|, \qquad |S|^* = \sup_{\sigma} \left| \sum_{\sigma} \xi_i \right|,$$

where σ is any subset of $1, 2, \cdots, p$. It is easily verified that $|S_1+S_2| = |S_1|+|S_2|$, $|S_1+S_2|^* \leqq |S_1|^*+|S_2|^*$, $|cS| = |c|\,|S|$, and $|cS|^* = |c|\,|S|^*$. Define further, when $|S| \neq 0$,

$$G(S) = |S|^*/|S|.$$

Note that $0 < G(S) \leqq 1$. Let $g = \inf G(S)$, taken over all sequences $S \subset B$.

THEOREM.[2] *If $g=0$, unconditional convergence does not imply absolute convergence in B. If $g>0$, the two are equivalent.*

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Suggested to the author by Garrett Birkhoff, in conversation.

PROOF. (i) If $g=0$, we can find a sequence $\{S_n\}$ such that $G(S_n)<4^{-n}$. Put $c_n=2^n/|S_n|$, and $S_n'=c_nS_n$. Then $|S_n'|=c_n|S_n|=2^n$, and $|S_n'|^*<4^{-n}|S_n'|=2^{-n}$. A well known necessary and sufficient condition for unconditional convergence of a series $\sum x_n$ is that given $\epsilon>0$, we can find N such that $|\sum_s x_n|<\epsilon$, where s is any finite set of integers, all greater than N. From this it is easily seen that the series formed by summing the elements of $S_1'+S_2'+S_3'+\cdots$ converges unconditionally but not absolutely.

(ii) If $g>0$, we may write $|S|\leqq|S|^*/g$, and it is now easily seen that unconditional convergence implies absolute convergence, and so the two are equivalent.

COROLLARY 1. *Unconditional convergence does not imply absolute convergence in L.*

PROOF. Let $r_n(t)$ denote the Rademacher orthogonal functions on $0\leqq t\leqq 1$, defined by $r_n(t)=2e_n(t)-1$, where $e_n(t)$ is the nth digit in the binary expansion of t, the value 0 being assigned at points that have two such expansions. Thus $r_n(t)$ takes the values -1 and $+1$ alternately on intervals of length $1/2^n$.

We shall show that if $S_n=(r_1, r_2, \cdots, r_n)$, we have

$$|S_n|^*/|S_n|\to 0, \tag{1}$$

from which the desired result follows by the theorem. Evidently $|S_n|=n$. Moreover, when σ is a subset containing p of the numbers $1, 2, \cdots, n$, use of the Schwarz inequality and the orthogonality of the functions $r_k(t)$ gives

$$\int_0^1\left|\sum_\sigma r_k(t)\right|dt\leqq\left\{\int_0^1\left|\sum_\sigma r_k(t)\right|^2dt\right\}^{1/2}=p^{1/2}.$$

Hence, since $p\leqq n$, $|S_n|^*\leqq n^{1/2}$. Therefore (1) holds, and the proof is complete.

COROLLARY 2. *Unconditional convergence does not imply absolute convergence in l.*

PROOF. Since there is no necessity for the sets S_n to have common elements, we may write

$$\begin{aligned} S_1 &= \{(-1,\ 1,\ 0,\ 0,\ 0,\cdots)\},\\ S_2 &= \{(-1,-1,\ 1,\ 1,\ 0,\ 0,\ 0,\cdots),\\ &\qquad (-1,\ 1,-1,\ 1,\ 0,\ 0,\ 0,\cdots)\},\end{aligned}$$

$$S_3 = \{(-1, -1, -1, -1, 1, 1, 1, 1, 0, 0, 0, \cdots),$$
$$(-1, -1, 1, 1, -1, -1, 1, 1, 0, 0, 0, \cdots),$$
$$(-1, 1, -1, 1, -1, 1, -1, 1, 0, 0, 0, \cdots)\},$$

and so on. Then $|S_n| = 2^n n$. Denote the sequences which make up S_n, taken in the order indicated, by $R_1^n, R_2^n, \cdots, R_n^n$, and denote the mth term of R_k^n by $R_k^n(m)$. Then

$$\sum_{m=1}^{\infty} \left| \sum_{k\in\sigma} R_k^n(m) \right| = 2^n \int_0^1 \left| \sum_{k\in\sigma} r_k(t) \right| dt \leqq 2^n n^{1/2},$$

as in Corollary 1. We then have $|S_n|^*/|S_n| \to 0$, and the result follows as before.

References

1. S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

2. S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Warsaw and Lwów, 1935.

3. W. Orlicz, *Über unbedingte Konvergenz in Funktionenräumen* (II), Studia Mathematica vol. 4 (1933) pp. 41–47.

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