SOME INVARIANTS OF CERTAIN PAIRS OF HYPERSURFACES

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Introduction. It is known [8, 9][1] that if two surfaces in ordinary space have a common tangent plane at an ordinary point, then the ratio of their total curvatures at this point is a projective invariant, and the theorem holds true similarly for hyperspaces.[2] In connection with this theorem and the investigation of Bouton [2], Buzano [3] and Bompiani [1] have shown the existence of a projective invariant, together with metric and projective characterizations, determined by the neighborhood of the second order of two surfaces S, S^* at two ordinary points O, O^* in ordinary space under the conditions that the tangent planes of the surfaces S, S^* at the points O, O^* be distinct and have OO^* for the common line. Furthermore, the other case in which the tangent planes of the surfaces S, S^* at the points O, O^* are coincident[3] has been considered in recent papers of the author [6, 7].

It is the purpose of the present paper to generalize the results of the two cases mentioned above.

Let V_{n-1}, V_{n-1}^* be two hypersurfaces in a space S_n of n dimensions, and t_{n-1}, t_{n-1}^* the tangent hyperplanes of the hypersurfaces V_{n-1}, V_{n-1}^* at two ordinary points O, O^*. For the subsequent discussion it is convenient to assume in Chapter I that the tangent hyperplanes t_{n-1}, t_{n-1}^* are coincident. We can (§1), as in ordinary space, determine a projective invariant by the neighborhood of the second order of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*; and the projective and metric characterizations of this invariant are given in the next two sections.

Chapter II treats of the case in which the tangent hyperplanes t_{n-1}, t_{n-1}^* are distinct, and the common tangent flat space t_{n-2} of t_{n-1}, t_{n-1}^* contains the line OO^*. We first (§4) show by analysis the existence of two projective invariants determined by the neighbor-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] The simple projective characterizations of this invariant were given by C. Segre [10] for two plane curves and by P. Buzano [4] for two surfaces in space S_n $(n>2)$. On the other hand, A. Terracini [11] also interpreted projectively this invariant by virtue of the conception of density of dualistic correspondences.

[3] It should be noted that for two plane curves having a common tangent at two ordinary points no projective invariant can be determined by the neighborhood of the second order of the two curves at these points. See my paper [5].

hood of the second order of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*; and then (§§5, 6) give them simple projective and metric characterizations. From the fact that one of the two invariants is reduced to 1 when the immersed space S_n is of three dimensions, it follows that our result in this chapter stands actually for a generalization of that of Buzano and Bompiani.

Chapter I. Two hypersurfaces with common tangent hyperplane at two ordinary points

1. **Derivation of an invariant.** Let V_{n-1}, V_{n-1}^* be two hypersurfaces in a space S_n of n dimensions with common tangent hyperplane t_{n-1} at two ordinary points O, O^*. Let $x_1, \cdots, x_{n+1}$ denote the homogeneous projective coordinates of a point in the space S_n. If we choose the points O, O^* to be the vertices $(1, 0, \cdots, 0)$, $(0, \cdots, 0, 1, 0)$ of the system of reference, and the common tangent hyperplane t_{n-1} to be the coordinate hyperplane $x_{n+1}=0$ of the system, then the power series expansions of the hypersurfaces V_{n-1}, V_{n-1}^* in the neighborhood of the points O, O^* may be written in the form

$$V_{n-1}:\quad \frac{x_{n+1}}{x_1} = \sum_{i,k=2}^{n} l_{ik}\frac{x_i}{x_1}\frac{x_k}{x_1} + \cdots, \tag{1}$$

$$V_{n-1}^*:\quad \frac{x_{n+1}}{x_n} = \sum_{i,k=1}^{n-1} m_{ik}\frac{x_i}{x_n}\frac{x_k}{x_n} + \cdots. \tag{2}$$

In order to find a projective invariant of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*, we have to consider the most general projective transformation of coordinates which shall leave the points O, O^* and the hyperplane t_{n-1} unchanged:

$$\begin{aligned} x_i &= \sum_{r=1}^{n+1} a_{ir}x_r' \qquad (i = 1, \cdots, n), \\ x_{n+1} &= a_{n+1,n+1}x_{n+1}', \end{aligned} \tag{3}$$

where

$$a_{21} = \cdots = a_{n1} = 0, \qquad a_{1n} = \cdots = a_{n-1,n} = 0, \tag{4}$$

$$D = \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix} \neq 0. \tag{5}$$

The effect of this transformation on equations (1), (2) is to produce two other equations of the same form whose coefficients, indicated by accents, are given by the formulas

$$
(6)\qquad
\begin{aligned}
a_{11}a_{n+1,n+1}l'_{ik} &= \sum_{r,s=2}^{n} a_{ri}a_{sk}l_{rs} &\qquad (i,\ k &= 2, \cdots, n),\\
a_{nn}a_{n+1,n+1}m'_{ik} &= \sum_{r,s=1}^{n-1} a_{ri}a_{sk}m_{rs} &\qquad (i,\ k &= 1, \cdots, n-1).
\end{aligned}
$$

From equations (4), (5), (6) it is easily seen that the determinants

$$
L = \begin{vmatrix} l_{22} & l_{23} \cdots & l_{2n} \\ l_{32} & l_{33} \cdots & l_{3n} \\ \cdot & \cdot \quad \cdot & \cdot \\ l_{n2} & l_{n3} \cdots & l_{nn} \end{vmatrix}, \qquad
M = \begin{vmatrix} m_{11} & m_{12} & \cdots m_{1,n-1} \\ m_{21} & m_{22} & \cdots m_{2,n-1} \\ \cdot & \cdot & \cdot \\ m_{n-1,1} & m_{n-1,2} & \cdots m_{n-1,n-1} \end{vmatrix},
$$

and their transformed ones L', M' are connected by the relations

$$
(7)\qquad
\begin{aligned}
a_{11}^{n-1}a_{n+1,n+1}^{n-1}L' &= a_{nn}^{2}D^{2}L,\\
a_{nn}^{n-1}a_{n+1,n+1}^{n-1}M' &= a_{11}^{2}D^{2}M.
\end{aligned}
$$

Further elimination of a_{ik} from equations (6), (7) shows immediately that *the quantity*

$$
(8)\qquad I = \frac{L}{M}\left(\frac{m_{11}}{l_{nn}}\right)^{(n+1)/3}
$$

is a projective invariant determined by the neighborhood of the second order of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*.

2. A projective characterization of the invariant I. Let the polar spaces of the line OO^* with respect to the asymptotic hypercones of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^* be respectively denoted by t_{n-2}, t_{n-2}^*, which determine a space t_{n-3} of $n-3$ dimensions in the common tangent hyperplane $x_{n+1}=0$. If the $n-2$ vertices, other than O and O^*, of the system of reference in the hyperplane $x_{n+1}=0$ be chosen in the space t_{n-3}, then the invariant I may be reduced to

$$
(9)\qquad I = \frac{L_{nn}}{M_{11}}\left(\frac{m_{11}}{l_{nn}}\right)^{(n-2)/3},
$$

where L_{nn}, M_{11} are the minors of l_{nn}, m_{11} in the determinants L, M respectively.

For the purpose of finding a projective characterization of the invariant I we first observe the space S_3 determined by the vertices $(1, 0, \cdots, 0)$, $(0, \cdots, 0, 1, 0)$, $(0, \cdots, 0, 1)$ and any one, say for instance $O_2(0, 1, 0, \cdots, 0)$, of the system of reference in the space t_{n-3}. The space S_3 intersects the hypersurfaces V_{n-1}, V_{n-1}^* in two surfaces S, S^*. Since the tangent planes of the surfaces S, S^* at the points O, O^* are coincident we have a projective invariant, denoted by J,

$$J = \frac{l_{22}}{m_{22}}\left(\frac{m_{11}}{l_{nn}}\right)^{1/3}, \tag{10}$$

whose projective characterization has been obtained [6].

Let Q (Q^*) be any quadric in the space S_3 which has OO_2 (O^*O_2), OO^* (OO^*) for generators and whose curve of intersection with the element of the second order of the surface S (S^*) at the point O (O^*) has a cusp at O (O^*). If the cone projecting from the point O_2 the curve of intersection of the two quadrics Q, Q^* be tangent to the common tangent plane OO^*O_2 along a line through the point O_2, then this line must be one of the lines (cf. [6])

$$\begin{gathered} x_n \pm (\pm 1)^{1/2}\left(\frac{m_{11}m_{22}}{l_{22}l_{nn}}\right)^{1/4} x_1 = 0, \\ x_3 = \cdots = x_{n-1} = x_{n+1} = 0. \end{gathered} \tag{11}$$

We may now uniquely determine a point P on the line OO^* such that the cross ratio of the three points O, O^*, P, and the intersection of the line (11) with OO^* is equal to $J^{1/4}$. On the other hand, the asymptotic hypercones of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^* determine a pencil of hyperquadrics in the hyperplane $x_{n+1}=0$, among which there exist n hypercones, two of them being the asymptotic hypercones. The line OO^* intersects each of the other $n-2$ hypercones in a pair of points. Let Q_i ($i=1, \cdots, n-2$) be any one of each pair of these points and D_i the cross ratio of the four points O, O^*, Q_i, P on the line OO^*, then we may easily show that *the invariant I can be expressed in terms of the $n-2$ cross ratios $D_1, D_2, \cdots, D_{n-2}$ as follows*:

$$I = (\pm 1)^{n-2}(D_1D_2 \cdots D_{n-2})^2. \tag{12}$$

3. A metric characterization of the invariant I. It is deemed worth while to give in this section a simple metric characterization of the invariant I. For this purpose we choose an orthogonal Cartesian coordinate system in such a way that the point O be the origin, the line OO^* be the X_{n-1}-axis, and the common tangent hyperplane t_{n-1} be the coordinate hyperplane $X_n=0$. Then the power series expansions of the hypersurfaces V_{n-1}, V^*_{n-1} in the neighborhood of the points O, O^* may be put into the form

$$(13)\qquad V_{n-1}:\ X_n = \sum_{i,k=1}^{n-1} \lambda_{ik}X_iX_k + \cdots,$$

$$(14)\qquad \begin{aligned} V^*_{n-1}:\ X_n = &\sum_{i,k=1}^{n-2} \mu_{ik}X_iX_k + 2\sum_{i=1}^{n-2} \mu_{i,n-1}X_i(X_{n-1} - h) \\ &+ \mu_{n-1,n-1}(X_{n-1} - h)^2 + \cdots, \end{aligned}$$

where h is the distance between the points O, O^*.

Let $y_0, y_1, \cdots, y_n$ be the homogeneous coordinates of a point defined by the formulas

$$(15)\qquad X_i = y_i/y_0 \qquad (i = 1, \cdots, n),$$

and let us consider the most general projective transformation of coordinates which shall leave the point O and the common tangent hyperplane t_{n-1} invariant, and change the point O^* into the vertex $(0, \cdots, 0, 1, 0)$ of the new coordinate system:

$$(16)\qquad \begin{aligned} y_0 &= \sum_{i=0}^{n} a_{0i}y_i', \\ y_i &= \sum_{r=1}^{n} a_{ir}y_r' \qquad (i = 1, \cdots, n-1), \\ y_n &= a_{nn}y_n', \end{aligned}$$

where

$$(17)\qquad a_{1,n-1} = \cdots = a_{n-2,n-1} = 0, \qquad a_{n-1,n-1} = ha_{0,n-1},$$

$$(18)\qquad \Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-2} \\ a_{21} & a_{22} & \cdots & a_{2,n-2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-2} \end{vmatrix} \neq 0.$$

By transformations (15) and (16), equations (13), (14) shall be carried into two others of the form

$$(19)\qquad V_{n-1}:\ \frac{y_n'}{y_0'} = \sum_{i,k=1}^{n-1} p_{ik}\frac{y_i'}{y_0'}\,\frac{y_k'}{y_0'} + \cdots,$$

$$(20)\qquad V^*_{n-1}:\ \frac{y_n'}{y_{n-1}'} = \sum_{i,k=0}^{n-2} q_{ik}\frac{y_i'}{y_{n-1}'}\,\frac{y_k'}{y_{n-1}'} + \cdots,$$

where the coefficients p_{ik}, q_{ik} are given by the equations:

$$(21)\qquad a_{00}a_{nn}p_{ik} = \sum_{r,s=1}^{n-1} a_{ri}a_{sk}\lambda_{rs} \qquad (i,\ k = 1, \cdots, n-1);$$

$$(22)\qquad a_{nn}a_{0,n-1}q_{ik} = \sum_{r,s=0}^{n-2} \alpha_{ri}\alpha_{sk}\mu_{rs} \qquad (i,\ k = 0, 1, \cdots, n-2),$$

$$(23)\qquad \begin{aligned} &\alpha_{00} = -ha_{00}, \quad \alpha_{i0} = 0, \quad \alpha_{0i} = a_{n-1,i} - ha_{0i}, \quad \alpha_{ri} = a_{ri}, \\ &\mu_{00} = \mu_{n-1,n-1}, \ \mu_{0r} = \mu_{r0} = \mu_{n-1,r} = \mu_{r,n-1} \ (i,\ r = 1, \cdots, n-2). \end{aligned}$$

Let

$$\Phi = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \cdots \lambda_{1,n-1} \\ \lambda_{21} & \lambda_{22} & \cdots \lambda_{2,n-1} \\ \cdot & \cdot & \cdot \\ \lambda_{n-1,1} & \lambda_{n-1,2} & \cdots \lambda_{n-1,n-1} \end{vmatrix}, \quad \Psi = \begin{vmatrix} \mu_{11} & \mu_{12} & \cdots \mu_{1,n-1} \\ \mu_{21} & \mu_{22} & \cdots \mu_{2,n-1} \\ \cdot & \cdot & \cdot \\ \mu_{n-1,1} & \mu_{n-1,2} & \cdots \mu_{n-1,n-1} \end{vmatrix},$$

$$P = \begin{vmatrix} p_{11} & p_{12} & \cdots p_{1,n-1} \\ p_{21} & p_{22} & \cdots p_{2,n-1} \\ \cdot & \cdot & \cdot \\ p_{n-1,1} & p_{n-1,2} & \cdots p_{n-1,n-1} \end{vmatrix}, \quad Q = \begin{vmatrix} q_{00} & q_{01} & \cdots q_{0,n-2} \\ q_{10} & q_{11} & \cdots q_{1,n-2} \\ \cdot & \cdot & \cdot \\ q_{n-2,0} & q_{n-2,1} & \cdots q_{n-2,n-2} \end{vmatrix},$$

then from equations (17), (18), (21), (22), (23) we obtain

$$(24)\qquad a_{00}^{n-1}a_{nn}^{n-1}P = a_{n-1,n-1}^{2}\Delta^{2}\Phi, \qquad a_{nn}^{n-1}a_{0,n-1}^{n-1}Q = h^{2}a_{00}^{2}\Delta^{2}\Psi.$$

Making use of the result obtained in §1 and observing equations (19), (20) we see that the projective invariant I associated with the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^* is

$$(25)\qquad I = \frac{P}{Q}\left(\frac{q_{00}}{p_{n-1,n-1}}\right)^{(n+1)/8}.$$

Furthermore, substituting (21), (22), (24) in (25) and reducing by equations (17) it follows that *the invariant I now takes the form*

$$(26)\qquad I = \frac{\Phi}{\Psi}\left(\frac{\mu_{n-1,n-1}}{\lambda_{n-1,n-1}}\right)^{(n+1)/8}.$$

Let K, K^* be the curvatures of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*; and R, R^* the curvatures at the points O, O^* of the plane sections of the hypersurfaces V_{n-1}, V_{n-1}^* made by the plane of the line OO^* and the normal to the common tangent hyperplane t_{n-1} at any point on the line OO^*. By a known formula it is easy to demonstrate that

$$K/K^* = \Phi/\Psi, \qquad R/R^* = \lambda_{n-1,n-1}/\mu_{n-1,n-1}, \tag{27}$$

and therefore that

$$I = \frac{K}{K^*}\left(\frac{R^*}{R}\right)^{(n+1)/3}. \tag{28}$$

Hence we have the following theorem.

THEOREM. *Let V_{n-1}, V_{n-1}^* be two hypersurfaces in a space S_n of n dimensions having a common tangent hyperplane t_{n-1} at two ordinary point O, O^*; K, K^* the curvatures of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*; and R, R^* the curvatures at the points O, O^* of the plane sections of the hypersurfaces V_{n-1}, V_{n-1}^* made by the plane of the line OO^* and the normal to the common tangent hyperplane t_{n-1} at any point on the line OO^*. Then $(K/K^*)(R^*/R)^{(n+1)/3}$ is a projective invariant associated with the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*.*

CHAPTER II. TWO HYPERSURFACES WITH DISTINCT TANGENT HYPERPLANES AT TWO ORDINARY POINTS

4. **Derivation of invariants.** Let V_{n-1}, V_{n-1}^* be two hypersurfaces in a space S_n of n dimensions such that the tangent hyperplanes t_{n-1}, t_{n-1}^* at two ordinary points O, O^* are distinct, and the common tangent flat space t_{n-2} of t_{n-1}, t_{n-1}^* contains the line OO^*. If we choose the points O, O^* to be the vertices $(0, 1, 0, \cdots, 0)$, $(0, \cdots, 0, 1, 0)$ of a homogeneous projective coordinate system of reference, and the tangent hyperplanes t_{n-1}, t_{n-1}^* to be the coordinate hyperplanes $x_1 = 0$, $x_{n+1} = 0$ respectively, then the power series expansions of the hypersurfaces V_{n-1}, V_{n-1}^* in the neighborhood of the points O, O^* may be written in the form

$$V_{n-1}: \quad \frac{x_1}{x_2} = \sum_{i,k=3}^{n+1} l_{ik}\frac{x_i}{x_2}\frac{x_k}{x_2} + \cdots, \tag{29}$$

$$V_{n-1}^*: \quad \frac{x_{n+1}}{x_n} = \sum_{i,k=1}^{n-1} m_{ik}\frac{x_i}{x_n}\frac{x_k}{x_n} + \cdots. \tag{30}$$

Considering the most general projective transformation of coordinates which shall leave the points O, O^* and the hyperplanes t_{n-1}, t_{n-1}^* unchanged, we may easily show as in §1 that *the quantities*

$$I = \frac{LMl_{nn}m_{22}}{L_{n+1,n+1}M_{11}}, \qquad J = \left(\frac{M}{L}\right)^{n-3}\left(\frac{L_{n+1,n+1}m_{22}}{M_{11}l_{nn}}\right)^{n+1} \tag{31}$$

are projective invariants determined by the neighborhood of the second order of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*, where $L_{n+1,n+1}$, M_{11} are respectively the minors of $l_{n+1,n+1}$, m_{11} in the determinants

$$L = \begin{vmatrix} l_{33} & l_{34} & \cdots & l_{3,n+1} \\ l_{43} & l_{44} & \cdots & l_{4,n+1} \\ \cdot & \cdot & \cdot & \cdot \\ l_{n+1,3} & l_{n+1,4} & \cdots & l_{n+1,n+1} \end{vmatrix}, \quad M = \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1,n-1} \\ m_{21} & m_{22} & \cdots & m_{2,n-1} \\ \cdot & \cdot & \cdot & \cdot \\ m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,n-1} \end{vmatrix},$$

and L', M', $L'_{n+1,n+1}$, M'_{11} are denoted by similar expressions.

5. **Projective characterizations of the invariants I, J.** By suitable choice of the system of reference the invariants I, J of equations (31) can be simplified. In fact, if we choose $n-1$ vertices of the system in the common tangent flat space t_{n-2}, and the other two $O_{n+1}(0, \cdots, 0, 1)$, $O_1(1, 0, \cdots, 0)$ respectively on the polars t, t^* of the flat space t_{n-2} with respect to the asymptotic hypercones of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*, the invariants I, J then take the simple form

$$I = l_{nn}l_{n+1,n+1}m_{11}m_{22},$$
$$J = \left(\frac{L_{n+1,n+1}}{M_{11}}\right)^4 \left(\frac{m_{11}}{l_{n+1,n+1}}\right)^{n-3} \left(\frac{m_{22}}{l_{nn}}\right)^{n+1}. \tag{32}$$

It should be noticed that the invariant J is reduced to 1 as $n=3$.

The polars t, t^* determine a space S_3, which intersects the hypersurfaces V_{n-1}, V_{n-1}^* in two surfaces S, S^*. These two surfaces S, S^* are evidently in the class considered by Buzano and Bompiani, and the corresponding invariant may be easily found from Bompiani's note [1] to coincide just with the invariant I. Thus we reach the conclusion:

The invariant I associated with the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^* is the invariant of Buzano at the points O, O^* of the surfaces S, S^* in which the hypersurfaces V_{n-1}, V_{n-1}^* are intersected by the space S_3 determined by the polars t, t^*.

To characterize projectively the other invariant J we consider any hyperplane π_α through the common tangent flat space t_{n-2}:

$$x_{n+1} = \alpha x_1 \qquad (\alpha \neq 0), \tag{33}$$

which intersects the hypersurfaces V_{n-1}, V_{n-1}^* in two hypersurfaces V_{n-2}, V_{n-2}^* of $n-2$ dimensions. Since these two hypersurfaces V_{n-2}, V_{n-2}^* have a common tangent hyperplane at the points O, O^* we may determine an invariant, denoted by I_α, as in §1:

$$I = \alpha^{2(n-3)/3}\frac{L_{n+1,n+1}}{M_{11}}\left(\frac{m_{22}}{l_{nn}}\right)^{n/3}. \tag{34}$$

On the other hand, it is useful to consider the hypercones C, C^* projecting respectively from the vertices $O_1(1, 0, \cdots, 0)$, $O_{n+1}(0, \cdots, 0, 1)$ the asymptotic hypercones at the points O, O^* of the hypersurfaces V_{n-1}, V_{n-1}^*. These two hypercones C, C^* determine a pencil of hyperquadrics in the space S_n, among which there exist $n-1$ hypercones, two of them being C, C^*. The line O_1O_{n+1} intersects each of the other $n-3$ hypercones in a pair of points. Let Q_i $(i=1, \cdots, n-3)$ be any one of each pair of these points, P the point of intersection of the line O_1O_{n+1} with the hyperplane π_α, and D_i the cross ratio of the four points O_1, O_{n+1}, Q_i, P on the line O_1O_{n+1}; then it follows that *the invariant J can be expressed in terms of the invariant I_α and the $n-3$ cross ratios $D_1, D_2, \cdots, D_{n-3}$ as follows*:

$$J = I_\alpha^3(D_1D_2\cdots D_{n-3})^2. \tag{35}$$

6. **Metric characterizations of the invariants I, J.** For the purpose of finding simple metric characterizations of the invariants I, J, we choose an orthogonal Cartesian coordinate system in such a way that the point O is the origin, the line OO^* is the X_{n-1}-axis, and the tangent hyperplane t_{n-1} is the coordinate hyperplane $X_1=0$. Then the power series expansions of the hypersurfaces V_{n-1}, V_{n-1}^* in the neighborhood of the points O, O^* may be put into the form

$$V_{n-1}\colon\ X_1 = \sum_{i,k=2}^{n}\lambda_{ik}X_iX_k + \cdots, \tag{36}$$

$$\begin{aligned} V_{n-1}^*\colon\ X_n = \mu X_1 &+ \sum_{i,k=1}^{n-2}\mu_{ik}X_iX_k + 2\sum_{i=1}^{n-2}\mu_{i,n-1}X_i(X_{n-1}-h)\\ &+ \mu_{n-1,n-1}(X_{n-1}-h)^2 + \cdots, \end{aligned} \tag{37}$$

where h is the distance between the points O, O^*, and $\mu=\cot\omega$, ω being the angle of the tangent hyperplanes t_{n-1}, t_{n-1}^*.

In order to express the two invariants I, J in terms of the coefficients of expansions (36), (37) we have first as in §3 to consider the homogeneous coordinates $y_0, y_1, \cdots, y_n$ of a point defined by formulas (15) and the most general projective transformation of coordinates, which shall leave the point O and the tangent hyperplane t_{n-1} invariant and carry the point O^* and the tangent hyperplane t_{n-1}^* into the vertex $(0, \cdots, 0, 1, 0)$ and the coordinate hyperplane

$y_n' = 0$ of the new coordinate system respectively. An easy calculation, which shall be omitted here, suffices to demonstrate the result as follows:

$$(38)\qquad I = h^4 \frac{\Phi\Psi\lambda_{n-1,n-1}\mu_{n-1,n-1}}{\Phi_{nn}\Psi_{11}}, \quad J = \left(\frac{\Psi}{\Phi}\right)^{n-3}\left(\frac{\Phi_{nn}\mu_{n-1,n-1}}{\Psi_{11}\lambda_{n-1,n-1}}\right)^{n+1},$$

where Φ_{nn}, Ψ_{11} denote respectively the minors of λ_{nn}, μ_{11} in the determinants

$$\Phi = \begin{vmatrix} \lambda_{22} & \lambda_{23} \cdots \lambda_{2n} \\ \lambda_{32} & \lambda_{33} \cdots \lambda_{3n} \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \lambda_{n2} & \lambda_{n3} \cdots \lambda_{nn} \end{vmatrix}, \qquad \Psi = \begin{vmatrix} \mu_{11} & \mu_{12} & \cdots \mu_{1,n-1} \\ \mu_{21} & \mu_{22} & \cdots \mu_{2,n-1} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \mu_{n-1,1} & \mu_{n-1,2} & \cdots \mu_{n-1,n-1} \end{vmatrix}.$$

Finally, we shall make use of the normals ON, ON^* at the point O of the common tangent flat space t_{n-2} in the tangent hyperplanes t_{n-1}, t_{n-1}^*. Let K_2, K_2^* be respectively the curvatures at the points O, O^* of the plane sections of the hypersurfaces V_{n-1}, V_{n-1}^* made by the planes OO^*N^*, OO^*N. Further, let K_n, K_n^* be the curvatures of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*; and K_{n-1}, K_{n-1}^* the curvatures at the points O, O^* of the hypersurfaces V_{n-2}, V_{n-2}^* of $n-2$ dimensions in which the tangent hyperplanes t_{n-1}^*, t_{n-1} intersect the hypersurfaces V_{n-1}, V_{n-1}^* respectively. Then

$$\begin{aligned} K_n &= 2^{n-1}\Phi, & K_n^* &= 2^{n-1}(1+\mu^2)^{-(n+1)/2}\Psi, \\ (39)\qquad K_{n-1} &= 2^{n-2}(1+\mu^2)^{(n-2)/2}\Phi_{nn}, & K_{n-1}^* &= 2^{n-2}\Psi_{11}, \\ K_2 &= 2(1+\mu^2)^{1/2}\lambda_{n-1,n-1}, & K_2^* &= 2\mu_{n-1,n-1}, \end{aligned}$$

and hence we arrive at *the following metric characterizations of the invariants I, J*:

$$(40)\qquad I = \frac{h^4}{16}\,\frac{K_nK_n^*K_2K_2^*}{K_{n-1}K_{n-1}^*\sin^{2(n-1)}\omega}, \qquad J = \left(\frac{K_n^*}{K_n}\right)^{n-3}\left(\frac{K_{n-1}K_2^*}{K_{n-1}^*K_2}\right)^{n+1}.$$

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