ON STRONG SUMMABILITY OF A FOURIER SERIES

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Let $s_n(x) = a/2\sum_{\nu=1}(a_\nu \cos \nu x + b_\nu \sin \nu x)$ be the partial sum of the Fourier series of an integrable periodic function $f(t)$ of period 2π, and let $\phi(t) = \{f(x+t)+f(x-t)-2s\}/2$. We shall establish the following result (Hardy-Littlewood [1][1]).

THEOREM. *If*

$$\int_0^t |\phi(u)| \{1 + \log^+ |\phi(u)| \} du = o(t), \qquad \text{as } t \to 0, \tag{1}$$

then $\sum_{\nu=0}^n |s_\nu(x) - s|^2 = o(n \log \log n)$, *as* $n \to \infty$.

To prove this theorem we require the following lemmas.

LEMMA 1. *If*

$$\int_0^t |\phi(u)| du = o(t), \qquad \text{as } t \to 0, \tag{2}$$

then

$$\sum_{\nu=0}^n |s_\nu(x) - s|^2 = \frac{1}{\pi^2}\int_{1/n}^{\delta} \frac{\phi(t)}{t^2} dt \int_{1/n}^{t} \phi(u) \frac{\sin n(u-t)}{u-t} du + o(n).$$

PROOF. By (2), for $\nu \leqq n$,

$$s_\nu(x) - s = \frac{2}{\pi}\int_{1/n}^{\delta} \phi(t) \frac{\sin \nu t}{t} dt + o(1).$$

Hence

$$\begin{aligned} \sum_{\nu=0}^n |s_\nu(x) - s|^2 &= \frac{4}{\pi^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta} \frac{\phi(u)\phi(t)}{ut}\left\{\sum_{\nu=1}^n \sin \nu t \sin \nu u\right\} dudt + o(n) \\ &= \frac{2}{\pi^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta} \frac{\phi(u)\phi(t)}{ut} \frac{\sin n(u-t)}{u-t} dudt \\ &\quad + \frac{2}{\pi^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta} \frac{\phi(u)\phi(t)}{ut} \frac{\sin n(u+t)}{u+t} dudt + o(n) \\ &= J_1 + J_2 + o(n). \end{aligned} \tag{3}$$

Received by the editors December 15, 1943.

[1] Numbers in brackets refer to the references listed at the end of the paper.

Set $\Phi(t) = \int_0^t |\phi(u)| du$. Then

$$J_2 = O\left\{ \int_{1/n}^{\delta} \frac{|\phi(t)|}{t^2} dt \int_{1/n}^{t} \frac{|\phi(u)|}{u} du \right\}.$$

But

$$\int_{1/n}^{\delta} \frac{|\phi(t)|}{t^2} dt \int_{1/n}^{\delta} \frac{|\phi(u)|}{u} du = O\left\{ \int_{1/n}^{\delta} \frac{|\phi(t)|}{t^2} \log ntdt \right\},$$

and

$$\int_{1/n}^{\delta} \frac{|\phi(t)|}{t^2} \log ntdt = \left[\frac{\Phi(t)}{t^2} \log nt \right]_{1/n}^{\delta} + 2 \int_{1/n}^{\delta} \frac{\Phi(t)}{t^3} \log ntdt$$

$$- \int_{1/n}^{\delta} \frac{\Phi(t)}{t^3} dt = o(n) + o\left(n \int_{1}^{\delta n} \frac{\log t}{t^2} dt \right)$$

$$+ o\left(\int_{1/n}^{\delta} \frac{dt}{t^2} dt \right) = o(n).$$

Hence

(4) $$J_2 = o(n).$$

By inversion of the order of integration, and the resolution of $1/t(t-u)$ into partial fractions, we get

(5)
$$\begin{aligned} J_1 &= \frac{2}{\pi^2} \int_{1/n}^{\delta} \frac{\phi(t)}{t^2} dt \int_{1/n}^{t} \phi(u) \frac{\sin n(n-t)}{u-t} du \\ &\quad + O\left(\int_{1/n}^{\delta} \frac{|\phi(t)|}{t^2} dt \int_{1/n}^{t} \frac{|\phi(u)|}{u} du \right) \\ &= \frac{1}{\pi^2} \int_{1/n}^{\delta} \frac{\phi(t)}{t^2} dt \int_{1/n}^{t} \phi(u) \frac{\sin n(u-t)}{u-t} du + o(n). \end{aligned}$$

Lemma 1 follows from (4) and (5).

LEMMA 2. *If $P(u)$ and $F(u)$ are non-negative integrable functions in (a, b), and $\psi(x)$ is a convex function in* $(0, \infty)$,

$$\psi\left\{ \int_a^b P(u)F(u)du \Big/ \int_a^b P(u)du \right\}$$

$$\leq \int_a^b P(u)\psi\{F(u)\} du \Big/ \int_a^b P(u)du.$$

This is the well known Jensen inequality (Hardy-Littlewood-Pólya [2]).

PROOF OF THE THEOREM. Since $1+\log^+|\phi(u)|\geqq 1$,

$$\int_0^t |\phi(u)|\,du \leqq \int_0^t |\phi(u)|\,\{1+\log^+|\phi(u)|\}\,du = o(t).$$

Hence by Lemma 1 it suffices to prove that

$$G_n = \int_{1/n}^{\delta} \frac{\phi(t)}{t^2}\,dt \int_{1/n}^{t} \phi(u)\,\frac{\sin n(u-t)}{u-t}\,du = o(n \log\log n).$$

Now

$$\begin{aligned} G_n &= \int_{1/n}^{n^{-1}\log n} \frac{\phi(t)}{t^2}\,dt \int_{1/n}^{t} \phi(u)\,\frac{\sin n(u-t)}{u-t}\,du \\ &\quad + \int_{n^{-1}\log n}^{\delta} \frac{\phi(t)}{t^2}\,dt \int_{1/n}^{t-n^{-1}\log n} \phi(u)\,\frac{\sin n(u-t)}{u-t}\,du \\ &\quad + \int_{n^{-1}\log n}^{\delta} \frac{\phi(t)}{t^2}\,dt \int_{t-n^{-1}\log n}^{t} \phi(u)\,\frac{\sin n(u-t)}{u-t}\,du \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{6}$$

By (2) we have

$$I_1 = \int_{1/n}^{n^{-1}\log n} \frac{|\phi(t)|}{t^2}\,O(nt)\,dt = o(n\log\log n), \tag{7}$$

and

$$I_2 = \int_{n^{-1}\log n}^{\delta} \frac{|\phi(t)|}{t^2}\,O\left(\frac{n}{\log n}\,t\right)dt = o(n). \tag{8}$$

Now set

$$P(u) = \left|\frac{\sin n(u-t)}{u-t}\right|,$$

$$k = \int_{t-n^{-1}\log n}^{t} P(u)\,du \sim \log\log n,$$

and

$$J_n(t) = \int_{t-n^{-1}\log n}^{t} |\phi(u)|\,P(u)\,du.$$

Applying Lemma 2 with $\psi(x)=x(1+\log^+ x)$, we find

$$\psi\left\{\frac{J_n(t)}{k}\right\} \leqq \frac{\int_{t-n^{-1}\log n}^{t} |\phi(u)|\,\{1+\log^+|\phi(u)|\,P(u)\,du}{k} \leqq \frac{nt}{k}\,.$$

Hence

$$J_n(t) \leqq nt\left\{1 + \log^+ \frac{J_n(t)}{k}\right\}^{-1}.$$

If we suppose now that

$$(9) \qquad J_n(t) \geqq A_1 nt/\log ntk^{-1},$$

we find $1+\log^+ (J_n(t)/k) \geqq A \log (nt/k)$, for $nt/k \geqq \log n/\log \log n$. Hence

$$(10) \qquad J_n(t) \leqq A_2 nt/\log ntk^{-1},$$

for all values of n and t such that (9) and the relation $nt \geqq \log n$ hold true.

If (9) does not hold,

$$(11) \qquad J_n(t) \leqq A_1 nt/\log ntk^{-1}.$$

By (10) and (11) we have then

$$(12) \qquad J_n(t) \leqq A_3 nt/\log ntk^{-1}$$

for all values of n and t for which $nt \geqq \log n$.

By (12)

$$\begin{aligned} |I_3| &\leqq \int_{n^{-1}\log n}^{\delta} \frac{|\phi(t)|}{t^2} J_n(t)dt \leqq A\int_{n^{-1}\log n}^{\delta} \frac{|\phi(t)|}{t^2} \frac{n}{\log (n/\log\log n)} dt \\ &\leqq An\left\{\left[\frac{\Phi(t)}{t} \frac{1}{\log (nt/\log\log n)}\right]_{n^{-1}\log n}^{\delta} \right. \\ &\quad + \int_{n^{-1}\log n}^{\delta} \frac{\Phi(t)}{t^2} \frac{dt}{\log (nt/\log \log n)} \\ &\quad \left. + \int_{n^{-1}\log n}^{\delta} \frac{\Phi(t)}{t^2} \frac{dt}{(\log (nt/\log\log n))^2}\right\} \\ &= o(n \log \log n). \end{aligned}$$

From this inequality and (6), (7), and (8) we draw

$$G_n = o(n \log \log n).$$

In conclusion I remark that by the above method I established in a recent paper the following result, which is a generalization of Hardy and Littlewood's theorem [1].

If

$$\int_0^t |\phi(u)|\, du = O(t) \quad \text{and} \quad \int_0^t \phi(u)du = o(t) \qquad \text{as } t \to 0,$$

then

$$\sum_{\nu=0}^{n} | s_\nu(x) - s |^2 = o(n \log n), \qquad \text{as } n \to \infty.$$

References

1. G. H. Hardy and J. E. Littlewood, *On strong summability for a Fourier series*, Fund. Math. vol. 25 (1935).
2. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, 1934, p. 151.

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