WILLIAM CASPAR GRAUSTEIN—IN MEMORIAM

William Caspar Graustein was born in Cambridge, Mass., November 15, 1888. He graduated from Harvard, Magna cum Laude in 1910. He spent the next year in the same place doing graduate work in mathematics, his main interest being then, as ever afterwards, in geometry. He received his first initiation into the higher branches of that subject by reading Klein's incomparable 1893 lectures on "Höhere Geometrie." As a matter of fact this nearly had a most disastrous effect, for when Study in Bonn heard that his prospective pupil had been reading Klein, he wrote, "So ist er ganz und gar verdorben." However, Study relented sufficiently to receive him in the autumn of 1911, thereby initiating a close personal and intellectual intimacy that lasted many years, and had the deepest influence on the whole of Graustein's subsequent career.

He took the doctor's degree, Summa cum Laude in July, 1913, and returned at once to Harvard as instructor in mathematics. He was then promoted to an assistant professorship in the newly formed Rice Institute in Houston, Texas. He joined his Cambridge friend Griffith Conrad Evans in building up there, with the warm approval of the President Edgar Odell Lovett, the most advanced mathematical centre south of Mason and Dixon's line. This lasted for four fruitful years, then came the war. Graustein was bound he would serve his country, in spite of official discouragement on account of his name and antecedents, and serve he did in the Ordnance Department at the proving ground at Aberdeen, Maryland, eventually becoming a first lieutenant.

He returned to Harvard in the autumn of 1919 to help in the congenial task of rehabilitating the Division of Mathematics, which in one year had lost three of its most valuable members through death and transfer. He worked mightily at that task till the day of his death. He was chairman of the Division from 1932 to 1937, and Assistant Dean of the Faculty of Arts and Sciences 1939–41. But he served mathematics outside his own university as well, acting for twelve years (1924–36) as an Associate Editor, and for five years (1936–41) as one of the Editors, of the Transactions of the American Mathematical Society, and chairman of the very busy organizing committee of the proposed 1940 International Mathematical Congress in 1938. He was killed in a motor accident, January 22, 1941.

Graustein's doctoral thesis, written with Study, was entitled "Eine reelle Abbildung analytischer komplexer Raumkurven." For a number of years Study had been interested in the geometry of the complex domain. In three-space we may have varieties depending analytically on any number from one to five of real parameters. But it is always heartening to have something real to represent complex elements, and many writers had occupied themselves with this problem of representation. Study had recently flirted with one of the methods, which was based on an idea initiated by Poncelet, but more closely associated with the writing of a vituperative Frenchman, F. Maximilien Marie. It consists in replacing each complex point by an ordered pair of real points, the closest pair of the elliptic involution whose double points are the given point and its conjugate. This conjugate is represented by the same pair in opposite order. Analytically the point with rectangular Cartesian coordinates x^i is represented by the ordered pair

$$y^i = \frac{x^i + \bar{x}^i}{2} + i\,\frac{x^i - \bar{x}^i}{2}; \qquad \omega^i = \frac{x^i + \bar{x}^i}{2} - i\,\frac{x^i - \bar{x}^i}{2}.$$

When it comes to representing simple loci like lines, circles and planes, there are a distressing number of special cases which must be handled separately; the isotropic elements cause trouble. The points of a complex space curve depend on two real parameters, and will be represented by the points of two real surfaces connected by an analytic relation. Graustein shows that:

A. They are translation surfaces.

B. There is a first and a second system of curves on the one so related to a first and a second system on the other that at corresponding points the tangent to the first curve on either surface is parallel to that of the second curve on the other.

C. The relation is area-preserving, but not conformal.

Graustein retained his interest in complex geometry for a number of years. In Volume 16 of the Transactions he takes up the question of when pairs of complex elements in euclidean three-space can be carried into one another by real motions. The criterion depends on certain real invariants, as the squared distance from a complex point to its conjugate. In the twenty-fourth volume of the same journal he generalizes the Marie representation. What is the most general representation where a complex point (z) is represented by a real ordered pair $(x)(y)$ it being required that the conjugate point $(\bar{z})$ shall be represented by $(y)(x)$, a real point shall be represented by itself counted twice, the functions involved shall be analytic, and the relations involved shall be invariant under some specified group. Two years later in Volume 26 of the Annals of Mathematics he relates the generalized method of representation to the main problem of his thesis.

A complex curve is represented by two real surfaces in one-to-one correspondence with correspondence of parallelism. This idea led Graustein to make a further study of the relation of two surfaces when the normals at corresponding points are parallel, and in Volume 23 of the Transactions he published a paper on "Parallel Maps of Surfaces." If we have a one-to-one correspondence between two surfaces, neither of which is developable, of such a sort that normals at corresponding points are parallel, they have the same spherical representation. If a point P' corresponds to a point P, there is a projective relation among the tangents at P' connecting each with that which is parallel to the tangent at P corresponding to the given one. When the relation is non-parabolic, that is to say, has two self-corresponding elements, the projective invariant I of this relation is fundamental in the transformation. Graustein shows that it is equal to the cross ratio which the pair PP' make with the corresponding focal points in the congruence of lines connecting corresponding points. This beautiful result leads to many interesting theorems. The author wrote five other short papers dealing with mapping and applicability.

The development of the tensor calculus of Ricci and Levi-Civita introduced a modification in differential geometry as profound as that produced nearly a century before by the publication of Gauss' "Disquisitiones Generales." Many geometers threw themselves entirely into the new work; Graustein was more cautious, he recognized the advantages in the new notations, new points of view and new techniques, especially when more than three dimensions were involved. But what attracted him most was the invariant or covariant character of the new processes, and that led him to the idea of developing methods on the more classical lines, which seemed to him to go really deeper into the geometrical questions involved. The first result of this study was his "Méthodes Invariantes dans la Géométrie Infinitésimale," crowned by the Belgian Academy of Sciences in 1929. A year later appeared in this Bulletin, Volume 36, an address delivered by request before this Society, "Invariant Methods in Classical Differential Geometry."

What sort of things are invariant under the transformations of differential geometry? First of all Jacobians are. A vector normal to a surface is not concerned with the choice of parametric curves on the surface, and the three components of such a vector are proportional to the three Jacobians $\partial(x^ix^j)/\partial(uv)$. Again we have three fundamental quadratic differential forms, that which gives the directions of the isotropic lines, that which gives the asymptotic lines, and that which gives the isotropic directions on the spherical representation. Their algebraic invariants will give geometric invariants of the surface. Beyond these there is an invariant process called directional differentiation. This was, I think, first developed by Lamé. Graustein uses it with telling effect. The directional derivative of a point function f in the direction of a given element of arc is

$$\lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} = \frac{\partial f}{\partial s}.$$

The great trouble is that the order of differentiation is material when we perform the process twice

$$\frac{\partial^2 f}{\partial s \partial s'} \neq \frac{\partial^2 f}{\partial s' \partial s}.$$

He turns the difficulty by introducing a new operator ∇. If s and s' are arcs on an orthogonal system of curves,

$$\frac{\nabla f}{\nabla s} = \frac{\partial f}{\partial s} + \frac{1}{\sigma'} f, \qquad \frac{\Delta f}{\Delta s'} = \frac{\partial f}{\partial s'} - \frac{1}{\sigma} f$$

where $1/\sigma$ is the geodesic curvature of s and $1/\sigma'$ that of s'. It then appears that

$$\frac{\nabla}{\nabla s} \frac{\partial f}{\partial s'} = \frac{\nabla}{\nabla s'} \frac{\partial f}{\partial s}.$$

The Gauss and Mainardi-Codazzi equations are easily found in terms of these new symbols.

It is clear that such invariant methods, though first developed for the case of euclidean measure in three-space, can be widely generalized. Graustein did just this in the thirty-sixth volume of the Transactions. Here we have a forty-page article on the geometry of Riemann spaces. The basis of directional differentiation is a mixed tensor given by the equation

$$\frac{\partial}{\partial s^q} \frac{\partial f}{\partial s^p} - \frac{\partial}{\partial s^p} \frac{\partial f}{\partial s^q} = B_{ij}k \frac{\partial f}{\partial s^k}.$$

Another development of these directional methods appeared two years earlier in the same journal, in an article entitled "Parallelism and Equidistance in Differential Geometry." Consider two curves C and $C' + \Delta C'$ on a surface. What shall we mean by saying that they are equidistant with regard to a one-parameter family of curves C'? Clearly that the arcs of the latter bounded by the first two are all equal. But when they are not equal we should like to develop some measure of their spreading. If the curves C are given by the values of the function f Graustein defines as the "distantial spread" of the curves C with regard to the curves C'

$$\frac{1}{b'} = -\frac{\partial}{\partial s}\log\left|\frac{\partial f}{\partial s'}\right|.$$

I mention in passing a personal regret that the author found it necessary to mar such a pretty conception with such a cacophonous name. Analogous to this is the angular spread, defined by the equation

$$\frac{1}{a} = \frac{1}{\rho} + \frac{d\omega}{ds}$$

where ω is the properly measured angle between that curve C on which the distances are measured and the various curves C' while $1/\rho$ is the geodesic curvature. The two types of spreads are closely connected, and have many pretty properties. $1/\rho$ vanishes in the case of Levi-Civita parallelism. The two are connected by interesting relations.

I will give it as my personal opinion that Graustein's work dealing with invariant methods was his most important contribution to mathematical science. His last published paper, in the forty-seventh volume of the Transactions, was somewhat different. It gave a complete and beautiful solution of the difficult problem of harmonic minimal surfaces. This had been only incompletely handled by previous writers.

Any notice of Graustein's life work should include some mention of his textbooks. The first of these, "Plane and Solid Analytic Geometry" written in collaboration with W. F. Osgood appeared in 1921. It is a thoroughly usable introductory text, with adequate exercises. It has never seemed to me, however, that the book was entirely at peace with itself. When a distinguished analyst and an enthusiastic geometer work together, if both are good teachers, the result may well be a good book, but it may also lack the inner consistency which it would have possessed had either author written it alone. No such charge can be brought against Graustein's "Introduction to Higher Geometry" which first saw the light in 1930 after many years of most careful preparation. Soon after W. E. Byerley returned to Harvard in 1876 he introduced a course in "Modern

Methods in Analytic Geometry" drawing its inspiration from the lectures by Evan W. Evans that he had heard while he was at Cornell. That course became at once one of the fixtures in the Harvard mathematical curriculum, successive teachers modified it to suit their own ideas and Graustein gave it repeatedly. This excellent book represented the results of his experience. The first three quarters are devoted to a careful presentation of the projective geometry of points, lines, and conic sections, combining geometric and algebraic methods easily and naturally. Then fifty pages are given to the circle, and the geometry of inversion. This is prettily done. I cannot help feeling, however, that it should have been postponed to the end of the book, as the author seems to have introduced it because he felt, and rightly, that no well trained geometer should be ignorant of these things, rather than because it fitted naturally with the rest of the work. The last chapter of forty pages, deals all too briefly with the projective geometry of three-dimensional space.

Graustein's last textbook "Differential Geometry" appeared in 1935. This also represents the results of many years of teaching an introductory course in the subject. It is classical and conservative in spirit, great use is made of vector methods, and Study's vector notation. Near the end we have a twenty page introduction to the absolute differential geometry. It is interesting to contrast this book with the recent one on the same subject by Eisenhart, published just five years after. One has the impression that Graustein introduced some ideas of absolute geometry as a matter of conscience, he did not like the idea that a student who had finished the book should have no knowledge whatever of the more modern technique. Eisenhart introduces tensor notation as soon as he can and feels himself cramped as long as he is confined to three dimensions.

It is not the function of this Bulletin to carry long reviews of textbooks, but it is eminently proper to bring to the memory of the members of our Society beautifully written works of a great teacher and great friend of all who teach or learn mathematics.

William C. Graustein

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J. L. Coolidge