V_m IN S_n WITH PLANAR POINTS ($m \geqq 3$)

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1. **Introduction.** In this paper we shall classify the m-dimensional Riemannian manifolds (V_m) which are imbedded in an n-dimensional space of constant curvature (S_n) and whose normal curvature locus consists solely of planar points ($m \geqq 3$). Under the assumption that the second fundamental tensors have principal directions, we easily prove Segre's theorem:* *V_m in S_n with axial points are V_m in S_{m+1} or have second fundamental tensor of rank one.* Our proof is not as general as Segre's since the above additional assumption is required. However, our method can be generalized to classify the V_m in S_n with planar points. This classification is accomplished by use of the ranks of any of the two second fundamental tensors, which determine the normal curvature locus, and certain of the Ricci vectors. Our principal result is: *If the rank of any of these second fundamental tensors is greater than two, then V_m in S_n with planar points are* (1) *V_m consisting of ∞^1 V_{m-1} imbedded in ∞^1 S_{m+1}; or* (2) *V_m consisting of ∞^1 V_{m-1} imbedded in ∞^1 S_m; or* (3) *V_m lying in S_{m+2}.*

2. **Notation.** Let the unit tangent vector fields of m mutually orthogonal nonisotropic congruences of V_m in S_n be denoted by

$$\underset{c}{i}^{\kappa} = \underset{c}{\epsilon}\overset{c}{i}^{\kappa}, \qquad \underset{c}{\epsilon} = \pm 1, \qquad \kappa, \lambda, \mu = 1, \cdots, n, \tag{2.1}$$
$$\underset{c}{i}^{\kappa}\overset{d}{i}_{\kappa} = \overset{d}{\underset{c}{\delta}}, \qquad a, b, c = 1, \cdots, m.$$

According to whether ϵ is $+1$ or -1, we say that i^κ is in the positive or negative quadric of directions, determined by the first fundamental tensor of S_n ($a_{\lambda\mu}$)

$$a_{\lambda\mu}\underset{c}{i}^{\lambda}\underset{c}{i}^{\mu} = \underset{c}{\epsilon}. \tag{2.2}$$

The subscript in (2.1) refers to the congruence (orthogonal index), the contravariant index κ to the S_n coordinate system, the δ to the Kronecker symbol. For the $(n-m)$ mutually orthogonal unit vectors in the local E_{n-m} which is perpendicular to the local tangent E_m of the V_m at a point P, we write

$$\underset{p}{i}^{\kappa} \qquad p, q, r = m+1, \cdots, n. \tag{2.3}$$

* Most of the references are to Schouten-Struik, *Einfuhrung in die Neueren Methoden der Differentialgeometrie*, vols. 1 and 2. Noordhoff, Groningen, Batavia. Hence we shall merely indicate volume and page number: vol. 2, pp. 96, 99.

We denote the first fundamental tensor of the V_m of rank m by

$$a'_{\lambda\mu} = \sum_c \epsilon \overset{c}{i}_\lambda \overset{c}{i}_\mu = \overset{c}{i}_\lambda \overset{c}{i}_\mu. \tag{2.4}$$

Hence, the connecting unit affinor of V_m with respect to S_n becomes

$$B_\lambda^\kappa = \sum_c \epsilon \overset{c}{i}{}^\kappa \overset{c}{i}_\lambda = \overset{c}{i}{}^\kappa \overset{c}{i}_\lambda, \qquad B_{\lambda\mu}^{\kappa\nu} = B_\lambda^\kappa B_\mu^\nu. \tag{2.5}$$

For the second fundamental affinors, we write

$$\underset{p}{h}_{\lambda\mu} = - B_{\lambda\mu}^{\alpha\beta} \nabla_\alpha \underset{p}{i}_\beta = \epsilon \overset{p}{h}_{\lambda\mu}, \tag{2.6}$$

where ∇_μ denotes covariant differentiation with respect to the metric of S_n. Hence the curvature affinor is

$$H_{\lambda\mu}^{\cdot\cdot\kappa} = \overset{p}{h}_{\lambda\mu} \underset{p}{i}{}^\kappa = B_{\lambda\mu}^{\alpha\beta} \nabla_\alpha B_\beta^\kappa, \tag{2.7}$$

and for the vectors entering into the Codazzi relations (Codazzi vectors) we write

$$\underset{p}{\overset{q}{v}}_\lambda = B_\lambda^\mu (\nabla_\mu \underset{p}{i}{}^\kappa) \overset{q}{i}_\kappa = - B_\lambda^\mu (\nabla_\mu \overset{q}{i}{}^\kappa) \underset{p}{i}_\lambda. \tag{2.8}$$

Then the Gauss, Codazzi, Ricci* relations for V_m in S_n can be written

$$K'_{\alpha\beta\lambda\mu} + \kappa (a'_{\alpha\lambda} a'_{\beta\mu} - a'_{\alpha\mu} a'_{\beta\lambda}) = (\underset{p}{h}_{\beta\lambda} \overset{p}{h}_{\alpha\mu} - \underset{p}{h}_{\alpha\lambda} \overset{p}{h}_{\beta\mu}), \tag{2.9}$$

$$\nabla'_{[\mu} \overset{p}{h}_{\lambda]\alpha} + \underset{q}{\overset{p}{v}}_{[\mu} \overset{q}{h}_{\lambda]\alpha} = 0, \tag{2.10}$$

$$\nabla'_{[\mu} \underset{q}{\overset{p}{v}}_{\lambda]} + \underset{r}{\overset{p}{v}}_{[\mu} \underset{q}{\overset{r}{v}}_{\lambda]} = \overset{p}{h}_{\alpha[\mu} \underset{q}{h}_{\lambda]}{}^{\cdot\alpha}. \tag{2.11}$$

where $K_{\alpha\beta\lambda\mu}$ denotes the Riemann-Christoffel affinor of V_m, κ the curvature of S_n, and ∇'_μ denotes covariant differentiation with respect to the metric of V_m.

3. **An imbedment theorem.** Struik† has shown that the necessary and sufficient conditions that V_m in S_n lie in a totally geodesic S_{m+k} of S_n are

$$\underset{u}{\overset{p}{v}}_\lambda = 0, \qquad p, q = m+1, \cdots, n, \tag{3.1}$$

$$\underset{u}{h}_{\mu\lambda} = 0, \qquad u, v = m+k+1, \cdots, n. \tag{3.2}$$

We shall show that a weaker form of (3.1) is ample. Let us divide the orthogonal indices as follows:

* Vol. 2, p. 130.

† Vol. 2, p. 150.

$$p, q, r = m + 1, \cdots, n,$$
$$(3.3) \qquad x, y, z = m + 1, \cdots, m + k,$$
$$u, v, w = m + k + 1, \cdots, n.$$

Now we shall prove the following theorem:

THEOREM. *The necessary and sufficient conditions that V_m in S_n lie in a totally geodesic S_{m+k} of S_n are that a set of $(n-m)$ mutually orthogonal vectors exist in the normal E_{n-m} such that*

$$(3.4) \qquad \overset{x}{\underset{u}{v}}_\lambda = 0,$$

$$(3.5) \qquad \underset{u}{h}_{\lambda\mu} = 0.$$

Consider the equations*

$$(3.6) \qquad D_\alpha \underset{u}{i}^\lambda = B^\mu_\alpha \nabla_\mu \underset{u}{i}^\lambda = - \underset{u}{h}_{\alpha\cdot}{}^\lambda + \overset{p}{\underset{u}{v}}_\alpha \underset{p}{i}^\lambda.$$

By transvecting with the various unit vectors, (3.6) becomes

$$(3.7) \qquad \underset{a}{i}^\alpha D_\alpha \underset{u}{i}^\lambda = (\text{terms in } \underset{1}{i}^\lambda, \cdots, \underset{m}{i}^\lambda) + (\overset{p}{\underset{u}{v}}_\alpha \underset{a}{i}^\alpha) \underset{p}{i}^\lambda.$$

Furthermore†

$$(3.8) \qquad \underset{a}{i}^\alpha D_\alpha \underset{c}{i}^\lambda = (\text{terms in } \underset{1}{i}^\lambda, \cdots, \underset{m}{i}^\lambda) + (\underset{a}{i}^\alpha \underset{c}{i}^\beta \underset{p}{h}_{\alpha\beta}) \underset{p}{i}^\lambda.$$

If (3.4) and (3.5) are valid, then from (3.7), (3.8), we find

$$(3.9) \qquad \underset{a}{i}^\alpha D_\alpha \underset{[1}{i}^{\lambda_1} \cdots \underset{m+k]}{i}^{\lambda_{m+k}} = \sigma \underset{[1}{i}^{\lambda_1} \cdots \underset{m+k]}{i}^{\lambda_{m+k}}.$$

Hence this $(m+k)$-vector determines a geodesic S_{m+k} in S_n.‡ Conversely assume V_m lies in a geodesic S_{m+k} in S_n, then (3.9) is valid. Hence one easily finds the conditions (3.4), (3.5).

4. **V_m in S_n with axial points.** We shall study the V_m in S_n with axial points. For these manifolds, the curvature affinor becomes

$$(4.1) \qquad H_{\mu\lambda}^{\cdot\cdot\nu} = \underset{n}{h}_{\mu\lambda} \overset{n}{i}^\nu.$$

That is

$$(4.2) \qquad \underset{u}{h}_{\mu\lambda} = 0, \qquad u, v = m + 1, \cdots, n - 1.$$

Hence the Codazzi relations (2.10) become, for $p = u$,

$$(4.3) \qquad \overset{u}{\underset{n}{v}}_{[\mu} \overset{n}{h}_{\lambda]\alpha} = 0.$$

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* Vol. 2, p. 130, (13.61).

† Vol. 2, p. 85, (10.5α when expanded).

‡ Vol. 2, p. 285, (13.1α). A proof could be furnished by use of vol. 1, p. 72, (6.28); vol. 1, p. 99, (10.14).

If these quantities are referred to the orthogonal congruences of V_m, then

$$\underset{n}{\overset{u}{v}}_\lambda = \underset{n}{\overset{u}{v}}_a \overset{a}{\dot{i}}_\lambda, \qquad a, b, c = 1, \cdots, m, \tag{4.4}$$

$$\overset{n}{h}_{\mu\lambda} = \overset{n}{h}_{ad} \overset{a}{\dot{i}}_\mu \overset{d}{\dot{i}}_\lambda. \tag{4.5}$$

Thus (4.3) becomes after simplification,

$$\overset{n}{h}_{ad} \underset{n}{\overset{u}{v}}_c = \overset{n}{h}_{cd} \underset{n}{\overset{u}{v}}_a, \qquad a \neq c. \tag{4.6}$$

If the following determinant equation has real roots and simple elementary divisors*

$$\left| \overset{n}{h}_{\mu\lambda} - \rho a'_{\mu\lambda} \right| = 0, \tag{4.7}$$

then m principal directions exist such that

$$\overset{n}{h}_{ad} = 0, \qquad a \neq d, \tag{4.8}$$

is valid for the congruences determining the principal directions. Referred to these congruences, (4.6) becomes

$$\overset{n}{h}_{dd} \underset{n}{\overset{u}{v}}_a = 0, \qquad a \neq d. \tag{4.9}$$

Thus, if the second fundamental tensor has a rank greater than one, we can take $d=1, 2$ and conclude that

$$\underset{n}{\overset{u}{v}}_a = 0, \qquad u = m+1, \cdots, n-1. \tag{4.10}$$

Hence, from (3.4), (3.5), these V_m lie in totally geodesic S_{m+1} of S_n. If the second fundamental tensor has rank one, then from the Gauss relations (2.9), the V_m are S_n. Furthermore, Struik† has shown in this case, from the Codazzi relations (2.10), that V_m is developable. Hence we have Struik's extension of Segre's theorem:

THEOREM. *If V_m in S_n has axial points and the rank of its second fundamental tensor is greater than one, then V_m lies in a geodesic S_{m+1}. If the rank of this tensor is one, then V_m is a developable S_m.*

5. **V_m in S_n with planar points.** Consider a V_m in S_n with planar points. Then we find similarly to (4.2)

$$\underset{u}{h}_{\lambda\mu} = 0, \qquad u, v = m+1, \cdots, n-2. \tag{5.1}$$

The equations corresponding to (4.6) are

$$\overset{n}{h}_{ad} \underset{n}{\overset{u}{v}}_c + \overset{n-1}{h}_{ad} \underset{n-1}{\overset{u}{v}}_c = \overset{n}{h}_{cd} \underset{n}{\overset{u}{v}}_a + \overset{n-1}{h}_{cd} \underset{n-1}{\overset{u}{v}}_a. \tag{5.2}$$

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* L. P. Eisenhart, *Riemannian Geometry*, Princeton, 1926, p. 110.

† Vol. 2, p. 150.

This system can be divided into two distinct types of equations: (1) a, d, c all differ; (2) a differs from c, a coincides with d. Using the congruences for which (4.8) is valid, we write (5.2) for the case where a, d, c all differ

$$\overset{n-1}{h}_{ad} \underset{n-1}{\overset{u}{v}}_c = \overset{n-1}{h}_{cd} \underset{n-1}{\overset{u}{v}}_a, \qquad u \neq n-1, n;\ a, d, c \text{ unequal.} \tag{5.3}$$

We drop the super- and sub-indices $(u, n-1)$ while we analyze (5.3). To facilitate our work, we divide the indices into three types

$$\begin{aligned} a, b, c &= 1, 2, \cdots, m, \\ p, q, r &= 1, 2, \cdots, k, \\ x, y, z &= k+1, \cdots, m. \end{aligned} \tag{5.4}$$

The solutions of (5.3) fall into classes. In the first case, none of v_a nor h_{cd} are zero. Hence, replacing c in (5.3) by b (hence $m \geqq 3$) and dividing the resulting equation into the original one, we find

$$v_c/v_b = h_{cd}/h_{bd}, \qquad c \neq d,\ b \neq d. \tag{5.5}$$

In the second case, some of components v_p are zero. Let us assume that there are k of these (see (5.4)). Then from (5.3), we find

$$h_{pc} = 0, \qquad p \neq c, \tag{5.6}$$

unless all v_c vanish. The question arises as to whether (h_{xy}) can vanish. An easy calculation of (5.3) shows that this implies either that v_x or v_y vanishes, unless all h_{cd} vanish. Hence, no other h_{cd} than those of (5.6) can vanish unless all h_{cd} vanish. Therefore, the general solution of (5.3) is

$$v_1 = v_2 = \cdots = v_k = h_{1c} = h_{2c} = \cdots = h_{kd} = 0, \tag{5.7}$$

and (5.5) is valid for the remaining v_x, h_{xy} unless (1) all v_c vanish or (2) all h_{cd} vanish. These last two are distinct solutions of (5.3).

The number k which denotes the number of zero components of the above Ricci vector is independent of the superscript u. For if k is associated with u and k' with u' and $k<k'$, then from (5.7) $h_{xy}=0$, $x \neq y$, $x, y > k$. But, by the above discussion, this is impossible. Hence the rank of this vector, which is k, is independent of u.

By studying the equations (5.5) and using the symmetry property of h_{cd}, we find the following solutions for case one:

$$\underset{n-1}{\overset{u}{v}}_c = \overset{uw}{\theta} \underset{n-1}{\overset{w}{v}}_c, \qquad u, w \neq n-1, n, \tag{5.8}$$

$$\overset{n-1}{h}_{cd} = \overset{u}{\theta} \underset{n-1}{\overset{u}{v}}_c \underset{n-1}{\overset{u}{v}}_d, \qquad c \neq d. \tag{5.9}$$

These equations are valid in case two. Hence they constitute the solution of (5.3) excepting the possibilities that v_c are all zero or h_{cd} (both of index $n-1$) are all zero for c not d. By treating the quantities of index n, we reach similar conclusions for their components formed with respect to the congruences of principal directions of the second fundamental tensor of index $(n-1)$.

If we write (5.2) for the case where a differs from c, a coincides with d, and h_{cd} (index n) vanishes for $c \neq d$, but h_{cd} and v_c (index $n-1$) do not all vanish, then from (5.9) we find

$$\overset{n}{h}_{dd}\underset{n}{\overset{u}{v}}_c + \overset{n-1}{h}_{dd}\underset{n-1}{\overset{u}{v}}_c = \overset{u}{\theta}\underset{n-1}{\overset{u}{v}}_c\underset{n-1}{v}_d\underset{n-1}{\overset{u}{v}}_d, \qquad d \neq c;\ u \neq n-1, n. \tag{5.10}$$

The solutions of (5.10) can be divided into two types: (1) the rank of $h_{\lambda\mu}$ (index n) is greater than two; (2) the rank of this tensor is less than or equal to two. In neither case may the vector v_λ (indices u, $n-1$) be a zero vector, or the two second fundamental tensors possess the same principal directions. Hence if we add to the above cases the possibilities that (3) the vector v_λ (indices u, $n-1$) is a zero vector, and (4) the two second fundamental tensors have the same principal directions, then we have listed the four types of solutions of (5.2). Evidently, the orthogonal index n may be replaced by $(n-1)$ in each of the above cases.

In this paper we shall not consider solutions of type (2). Furthermore, by use of (5.2), we find that if $h_{\lambda\mu}$ (index n) is of rank greater than two, then solutions of type (4) lead to axial points. Hence, we shall study solutions of types (1) and (3).

6. **Solutions of type** (1). If the rank of $h_{\lambda\mu}$ (index n) is greater than two, but its principal directions do not coincide with those of $h_{\lambda\mu}$ and v_λ (both $h_{\lambda\mu}$ and v_λ of index $n-1$) is not a zero vector, then the only solutions of (5.10) are

$$\underset{n}{\overset{u}{v}}_c = \overset{u}{\beta}\underset{n-1}{\overset{u}{v}}_c, \qquad u \neq n-1, n, \tag{6.1}$$

$$\overset{n-1}{h}_{dd} = \overset{u}{\theta}\underset{n-1}{\overset{u}{v}}_d\underset{n-1}{\overset{u}{v}}_d - \overset{u}{\beta}\overset{n}{h}_{dd}. \tag{6.2}$$

Combining this result with equations (5.8) and (5.9), we have the tensor equations

$$\underset{n-1}{\overset{u}{v}}_\lambda = \overset{uw}{\theta}\underset{n-1}{\overset{w}{v}}_\lambda, \tag{6.3}$$

$$\underset{n}{\overset{u}{v}}_\lambda = \overset{u}{\beta}\underset{n-1}{\overset{u}{v}}_\lambda, \tag{6.4}$$

$$\overset{n-1}{h}_{\lambda\mu} = \overset{u}{\theta}\underset{n-1}{\overset{u}{v}}_\lambda\underset{n-1}{\overset{u}{v}}_\mu - \overset{u}{\beta}\overset{u}{h}_{\lambda\mu}. \tag{6.5}$$

From the discussion of the solutions (case 1) of (5.3), it follows that the θ terms cannot vanish.

From the Ricci relations (2.11), we find

$$\nabla'_{[\mu}\overset{u}{\underset{n}{v}}_{\lambda]} + \overset{s}{\underset{n}{v}}_{[\mu}\overset{u}{\underset{s}{v}}_{\lambda]} = 0. \tag{6.6}$$

Let us denote the $(m-1)$ congruences of V_m which are orthogonal to v_λ (index n) by

$$\underset{'g}{i}^{\lambda}, \qquad 'g, 'h = '2, '3, \cdots, 'm. \tag{6.7}$$

By transvecting, (6.6) becomes, in virtue of (6.3), (6.4),

$$\underset{'g}{i}^{\lambda}\underset{'h}{i}^{\mu}\nabla'_{[\mu}\overset{u}{\underset{n}{v}}_{\lambda]} = 0. \tag{6.8}$$

Hence the congruence v^λ (index n) is V_{m-1} normal.

Let us denote the quantities of this V_{m-1} by barred letters. Then we choose the following normals to V_{m-1} in S_n:

$$\begin{aligned} &\underset{n}{\bar{i}}^{\lambda} = \rho(\beta \underset{n-1}{i}^{\lambda} - \underset{n}{i}^{\lambda}), \qquad \rho = (1+\beta^2)^{-1/2}; \qquad \underset{n+1}{\bar{i}}^{\lambda} = \rho(\underset{n-1}{i}^{\lambda} + \beta\underset{n}{i}^{\lambda}); \\ &\underset{n-1}{\bar{i}}^{\lambda} = \underset{'1}{i}^{\lambda} = \gamma\overset{u}{\underset{n}{v}}^{\lambda}, \qquad \gamma = (\epsilon a_{\lambda\mu}\overset{u}{\underset{n}{v}}^{\lambda}\overset{u}{\underset{n}{v}}^{\mu})^{-1/2}; \qquad \underset{u}{\bar{i}}^{\lambda} = \underset{u}{i}^{\lambda}. \end{aligned} \tag{6.9}$$

$U = m+1, \cdots, n-2$. Furthermore, let $\overline{B}_\mu^{\lambda}$ denote the connecting affinor of V_{m-1} in S_n

$$\overline{B}_\mu^{\lambda} \doteq \underset{'g}{i}^{\lambda}\overset{'g}{i}_{\mu}; \tag{6.10}$$

then the second fundamental tensors of V_{m-1} in S_n are

$$\underset{p}{\bar{h}}_{\lambda\mu} = \overline{B}_{\lambda\mu}^{\alpha\beta}\nabla_\alpha \underset{p}{\bar{i}}_{\lambda}, \qquad p = m+1, \cdots, n+1. \tag{6.11}$$

From (6.9), we find

$$\begin{aligned} \overset{n}{\bar{h}}_{\lambda\mu} &= \rho\overline{B}_{\lambda\mu}^{\alpha\beta}(\beta\overset{n-1}{h}_{\alpha\beta} - \overset{n}{h}_{\alpha\beta}), \\ \overset{n+1}{\bar{h}}_{\lambda\mu} &= \rho\overline{B}_{\lambda\mu}^{\alpha\beta}(\overset{n-1}{h}_{\alpha\beta} + \beta\overset{n}{h}_{\alpha\beta}), \\ \overset{n-1}{\bar{h}}_{\lambda\mu} &= \overline{B}_{\lambda\mu}^{\alpha\beta}\nabla_\alpha \underset{'1}{i}_{\beta}, \\ \overset{u}{\bar{h}}_{\lambda\mu} &= \overline{B}_{\lambda\mu}^{\alpha\beta}\overset{u}{h}_{\alpha\beta} = 0, \qquad u = m+1, \cdots, n-2. \end{aligned} \tag{6.12}$$

From (6.5), we see that the V_{m-1} component of

$$\overset{n-1}{h}_{\alpha\beta} + \beta\overset{n}{h}_{\alpha\beta}$$

is zero; hence

$$(6.13)\qquad \overset{n+1}{\bar{h}}_{\lambda\mu} = 0, \qquad \overset{n}{\bar{h}}_{\lambda\mu} = -\rho\overline{B}^{\alpha\beta}_{\lambda\mu}(\beta^2 + 1)\overset{n}{h}_{\alpha\beta}.$$

Thus these V_{m-1} have planar points. For their Ricci vectors which depend on $\bar{i}^\lambda$ (index $n-1$) we find

$$(6.14)\qquad \underset{n-1}{\overset{u}{\bar{v}}}_\lambda = -\overline{B}^\mu_\lambda(\nabla_\mu \overset{u\kappa}{\bar{i}})\underset{n-1}{\bar{i}}_\kappa, \qquad u = m+1, \cdots, n-1;\ u \neq n,$$

$$(6.15)\qquad \underset{n-1}{\overset{n+1}{\bar{v}}}_\lambda = -\overline{B}^\mu_\lambda(\nabla_\mu \overset{m+1}{\bar{i}^\kappa})\underset{n-1}{\bar{i}}_\kappa.$$

From the definition of this $\bar{i}^\lambda$ and (6.14), it follows that the vectors in (6.14) are combinations of the V_m components of $\nabla_\mu i_\lambda$ (i_λ of index u). However, these components are zero. Hence the vectors in (6.14) vanish. By expanding (6.15), we find from (6.9)

$$(6.16)\qquad \underset{n-1}{\overset{n+1}{\bar{v}}}_\lambda = -\rho\overline{B}^{\mu}_{\lambda}\underset{1}{i}^{\alpha}(\nabla_\mu \underset{n-1}{i}_\alpha + \beta\nabla_\mu \underset{n}{i}_\alpha) = -\rho\overline{B}^{\mu}_{\lambda}\underset{1}{i}^{\alpha}(\underset{n-1}{h}_{\mu\alpha} + \beta\underset{n}{h}_{\mu\alpha}).$$

But this vanishes in virtue of (6.5). Hence we conclude

$$(6.17)\qquad \underset{n-1}{\overset{u}{\bar{v}}}_\lambda = 0, \qquad u = m+1, \cdots, n+1;\ u \neq n.$$

These ∞^1 V_{m-1} have planar points which are of type (3).

7. **Solutions of type** (3). Solutions of type (3) are characterized by the equations

$$(7.1)\qquad \underset{n-1}{\overset{u}{v}}_\lambda = 0, \qquad u = m+1, \cdots, n-2;\ u \neq n-1, n.$$

Writing (5.2) for $a=d$, $a\neq c$, we find

$$(7.2)\qquad \overset{n}{h}_{dd}\underset{n}{\overset{u}{v}}_c = 0, \qquad d \neq c.$$

These equations coincide with (4.9). As in that section, we make the following conclusions:

(7.3) If $h_{\lambda\mu}$ (index n) is of rank greater than one, then $v_\lambda=0$ (index n). From (3.4), (3.5), these V_m lie in S_{m+2}.

(7.4) If $h_{\lambda\mu}$ (index n) is of rank one, then, say, $h_{11}\neq 0$, $v_1\neq 0$ (index n).

Since the rank of the $h_{\lambda\mu}$ (index n) of §6 is greater than one, we have, from (7.3), the theorem:

THEOREM. *The V_m in S_n of type* (1) *consist of $\infty^1 V_{m-1}$ imbedded in $\infty^1 S_{m+1}$.*

We now study (7.4). From the equations

$$(7.5)\qquad \overset{n}{h}_{\lambda\mu} = \overset{n}{h}_{11}\underset{1}{i}_\lambda\underset{1}{i}_\mu, \qquad \underset{n}{\overset{u}{v}}_\lambda = \overset{u}{\alpha}\underset{1}{i}_\lambda,$$

$$(7.6)\qquad \overset{n-1}{h}_{\lambda\mu} = h_{ab}\overset{a}{i}_\lambda\overset{b}{i}_\mu, \qquad \underset{n-1}{\overset{u}{v}}_\lambda = 0,$$

and the Ricci equation

$$(7.7)\qquad \nabla'_{[\mu} \overset{u}{\underset{n-1}{v}}{}_{\lambda]} + \overset{s}{\underset{n-1}{v}}{}_{[\mu} \overset{u}{\underset{s}{v}}{}_{\lambda]} = 0,$$

we find

$$(7.8)\qquad \overset{n}{\underset{n-1}{v}}{}_{[\mu} \overset{u}{\underset{n}{v}}{}_{\lambda]} = 0.$$

Hence

$$(7.9)\qquad \overset{n}{\underset{n-1}{v}}{}_{\lambda} = \phi \underset{1}{i}_{\lambda}.$$

Now, using the Ricci relation,

$$(7.10)\qquad \nabla'_{[\mu} \overset{n}{\underset{n-1}{v}}{}_{\lambda]} = \underset{n-1}{h}{}_{\alpha[\mu} \overset{n}{h}{}_{\lambda]}{}^{\alpha},$$

and (7.9), we find

$$(7.11)\qquad \underset{a}{i}^{\lambda} \underset{b}{i}^{\mu} \nabla'_{[\mu} \underset{1}{i}_{\lambda]} = 0, \qquad a, b = 2, 3, \cdots, m.$$

Hence the congruences which are orthogonal to i^{λ} (index 1) build V_{m-1}. Finally, we analyze the Codazzi relations

$$(7.12)\qquad \nabla'_{[\mu} \overset{n}{h}{}_{\lambda]\alpha} = \overset{n}{\underset{n-1}{v}}{}_{[\mu} \overset{n-1}{h}{}_{\lambda]\alpha}.$$

By use of (7.5), (7.6), (7.9), these equations become

$$(7.13)\qquad \underset{1}{i}_{\alpha} \underset{1}{i}_{[\lambda} \nabla'_{\mu]} \overset{n}{h}_{11} + \overset{n}{h}_{11} \underset{1}{i}_{[\lambda} \nabla'_{\mu]} \underset{1}{i}_{\alpha} + \overset{n}{h}_{11} \underset{1}{i}_{\alpha} \nabla'_{[\mu} \underset{1}{i}_{\lambda]} = \phi \overset{n-1}{h}_{ad} \underset{1}{i}_{[\lambda} \overset{a}{i}_{\mu]} \overset{d}{i}_{\alpha}.$$

By transvecting, we obtain

$$(7.14)\qquad \overset{n}{h}_{11} \underset{b}{i}^{\alpha} \underset{c}{i}^{\mu} \nabla'_{\mu} \underset{1}{i}_{\alpha} = \phi \overset{n-1}{h}_{cb}, \qquad c, b \neq 1.$$

Using the notation of §6 for the quantities of V_{m-1}, the equations (7.14) become

$$(7.15)\qquad \overset{n-1}{\bar{h}}{}_{\lambda\mu} = \bar{B}_{\lambda\mu}^{\alpha\beta} \nabla_{\alpha} \underset{1}{i}_{\beta} = \phi(\overset{n}{h}_{11})^{-1} \bar{B}_{\lambda\mu}^{\alpha\beta} \overset{n-1}{h}_{\alpha\beta}.$$

Furthermore, if we define

$$(7.16)\qquad \overset{n}{\bar{h}}{}_{\lambda\mu} = \bar{B}_{\lambda\mu}^{\alpha\beta} \nabla_{\alpha} \underset{n}{i}_{\beta}, \qquad \overset{n+1}{\bar{h}}{}_{\lambda\mu} = \bar{B}_{\lambda\mu}^{\alpha\beta} \nabla_{\alpha} \underset{n-1}{i}_{\beta},$$

then from (7.5)

$$(7.17)\qquad \overset{n}{\bar{h}}{}_{\lambda\mu} = \bar{B}_{\lambda\mu}^{\alpha\beta} \overset{n}{h}_{\alpha\beta} = 0,$$

$$(7.18)\qquad \overset{n+1}{\bar{h}}{}_{\lambda\mu} = \bar{B}_{\lambda\mu}^{\alpha\beta} \overset{n-1}{h}_{\alpha\beta}.$$

From (7.15), (7.17), (7.18), we conclude that these ∞^1 V_{m-1} contain axial points. Hence the theorem:

THEOREM. *The* V_m *in* S_n *of type* (3) *are either* V_m *in* S_{m+2} *or contain* ∞^1 V_{m-1} *with axial points.*

If the rank of $h_{\lambda\mu}$ (index $n-1$) is greater than two, then the rank of $\overline{B}^{\alpha\beta}_{\lambda\mu}h_{\alpha\beta}$ (h of index $n-1$) is greater than one. In this case, from Segre's theorem, the ∞^1 V_{m-1} lie in ∞^1 S_m. If the rank of $h_{\lambda\mu}$ (index $n-1$) is two and its nonzero *domain** *does not* contain the nonzero domain of $h_{\lambda\mu}$ (index n), then the same result is valid; if it does contain the nonzero domain of this $h_{\lambda\mu}$, then $\overline{B}^{\alpha\beta}_{\lambda\mu}h_{\alpha\beta}$ (index $n-1$) is of rank one. In this last case the ∞^1 V_{m-1} are ∞^1 developable S_{m-1}.

From §§6 and 7, we have the theorem:

THEOREM. *If the rank of any of the two second fundamental tensors is greater than two, then* V_m *in* S_n *with planar points are* (1) V_m *consisting of* ∞^1 V_{m-1} *imbedded in* ∞^1 S_{m+1}, *or* (2) V_m *consisting of* ∞^1 V_{m-1} *imbedded in* ∞^1 S_m, *or* (3) V_m *lying in* S_{m+2}.

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ON TRANSITIVE GROUPS THAT CONTAIN CERTAIN TRANSITIVE SUBGROUPS†

W. A. MANNING

If a simply transitive permutation group G of compound degree n contains a regular abelian subgroup H of order n, and if at least one Sylow subgroup of H is cyclic, G is imprimitive. The proof of this important theorem, due to Wielandt,‡ is remarkable for its brevity. But familiarity with certain preliminary theorems of Schur's§ is assumed. Unfortunately these theorems, as presented by Schur, do not appear to be as elementary as they really are. It seems, therefore, worth while to offer a complete proof of Wielandt's theorem that is elementary throughout, free from the theories of rings and representations, and based on the fundamental concept of the double coset, introduced by Cauchy|| in 1846. Some generalizations, too, can readily be made.

* Vol. 1, p. 19; German "Gebiet."

† Presented to the Society, December 29, 1938, under the title *A note on transitive groups with regular subgroups of the same degree.*

‡ H. Wielandt, Mathematische Zeitschrift, vol. 40 (1935), p. 582.

§ I. Schur, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1933, p. 598.

|| A. L. Cauchy, Comptes Rendus, vol. 22 (1846), p. 630.