ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

290. Walter Leighton: *Proper continued fractions.*

This paper presents a generalization of the regular continued fraction which includes the "continued cotangent" algorithm of Lehmer as a special case. It is shown that corresponding to any sequence of positive integers $a_1, a_2, a_3, \cdots$ there exists a unique *proper* continued fraction expansion of the form $b_0+K(a_n/b_n)$ for each real number y_0. Here b_0 is an integer and $b_1, b_2, b_3, \cdots$ are positive integers with $b_n \geqq a_n$. The generalizations of a number of theorems pertaining to the regular case are obtained, including a generalization of the famous Lagrange theorem on the representation of a quadratic surd. (Received June 3, 1939.)

291. A. N. Lowan: *On the problem of wave-motion for the wedge of an angle.*

The object of this paper is the integration of the nonhomogeneous differential equation of wave-motion for the two-dimensional domain bounded by the half planes $\theta=0$ and $\theta=\theta_0$. In Part I the displacement is prescribed over both boundary planes. In Part II the displacement is prescribed over one of the boundary planes while the gradient of the displacement is prescribed over the other. Finally in Part III the gradient of the displacement is prescribed over both boundary planes. The solution of the systems satisfied by the Laplace transform are obtained by means of Green's theorem in terms of the appropriate Green's functions for the operator ∇^2+k^2. In Parts I and III the required Green's functions are obtained from the integral representations of arbitrary functions in terms of the Green's functions in the theory of heat conduction for the same domain. The Green's function in Part II is obtained from that of Part I by a reflection of the given domain on the line $\theta=\theta_0$. The transition from the Laplace transform to the functions themselves offers no difficulty. (Received June 16, 1939.)

292. A. N. Lowan and Jack Laderman: *On the distribution of errors in nth tabular differences.*

The object of this paper is the determination of the distribution of errors in the nth tabular differences of a mathematical table under the assumption of a rectangular distribution of the errors in the entries of the table, the entries being given to k decimal places. This problem is solved with the aid of the powerful theorem which states that the characteristic function of the distribution of a sum of random variables is equal to the product of the characteristic functions of the distributions of the individual variables. The characteristic function of the desired distribution is readily obtained and the formal integral representation of the frequency function is obtained by the well known formula for the inversion of the Fourier transform. The frequency function is an even function, vanishing for $|x| \geqq 2n^{-1} \cdot 10^{-k}$ and is defined by different polynomials in different intervals. Explicit expressions for the frequency function are given for the cases $n=1$, $n=2$, and $n=3$. The results for $n=2$ have been applied to an analysis of the errors in the second tabular differences of the function $\mathrm{Ci}(x)$ computed by the New York City WPA Project for the Computation of Mathematical Tables. (Received June 23, 1939.)

293. R. P. Agnew: *On kernels of faltung transformations.*

The main theorems proved are of the following type: If a kernel $J(t)$ is such that the faltung $y(s)=\int_{-\infty}^{\infty}J(t)x(s+t)dt$ has property P for each function x in class X, then $J(t)$ must have property Q. In each case, X is a certain class of nonnegative integrable generalized step-functions. If P is the property that $y(s)$ exists for at least one s, then Q is the property $\int_u^{u+A}|J(t)|dt$ is a bounded function of u for each $A>0$. If P is the property $y(s)$ is integrable, then Q is the property $J(t)$ is integrable. If P is the property $y(s)$ is essentially bounded, then Q is the property $J(t)$ is essentially bounded. Each of these theorems is supplemented by a theorem of familiar type which asserts that if J has property Q then (i) y has property P for each integrable function x and (ii) a certain constant determined by J is the *bound* of the transformation, that is, the least constant M such that a constant (norm) determined by y is less than or equal to $M\int_{-\infty}^{\infty}|x(t)|dt$ for each integrable function x. (Received June 26, 1939.)

294. R. P. Agnew: *On translations of functions and sets.*

If a sequence $x_1(t), x_2(t), \cdots$ of complex-valued functions measurable over $-\infty<t<\infty$ is such that, for each real sequence $\lambda_1, \lambda_2, \cdots$ (i) $\lim_{n\to\infty} x_n(t-\lambda_n)=0$ for each t in some set C of positive measure $|C|$, then, for each $\delta>0$, (ii) $\sum_{n=1}^{\infty}\text{l.u.b.}\,|E_t\{h\leqq t\leqq h+1;\,|x_n(t)|\geqq\delta\}|<\infty$ where l.u.b. is taken over $-\infty<h<\infty$. Conversely if $x_1(t), x_2(t), \cdots$ is a sequence of measurable functions for which (ii) holds, then for each sequence $\lambda_1, \lambda_2, \cdots$ (i) holds for almost all t in $-\infty<t<\infty$. Some lemmas on translations of sets, which are somewhat more incisive than those required for proof of the direct theorem, are proved and discussed. (Received July 31, 1939.)

295. Warren Ambrose: *Some properties of measurable stochastic processes.*

Some general conditions are found for the existence of what Doob has called a measurable stochastic process (Transactions of this Society, vol. 42 (1937), pp. 107–140). (Received August 2, 1939.)

296. H. A. Arnold: *Defective groups.*

In the usual postulates for a group, the closure postulate is altered so that it is required that $a \cdot b$ be in the group whenever a and b are in the group and are different. The mark $a \cdot a$ may or may not be an element for any a. The structure of the resulting system is investigated together with the effect of weakening the postulates on inverses and the unit. Methods of completion to form a proper group are discussed. (Received August 2, 1939.)

297. W. L. Ayres: *Peano spaces as the theory of continuous images of intervals.*

The properties of Peano spaces have been deduced almost exclusively from other characterizations rather than the original definition as the image of a closed interval under a continuous mapping. As Peano continua appear in applications usually from this definition, it seems desirable to develop the theory from this definition alone, and the present paper presents such a development. In particular a new and fairly simple proof of the important cyclic connectivity theorem is obtained, depending only on the arc-wise local connectivity of the space and arc-wise connectivity of connected open sets, both of which are proved easily from the fundamental definition. (Received July 29, 1939.)

298. Reinhold Baer: *A Galois theory of linear systems over commutative fields.*

N. Jacobson has recently succeeded in extending Galois' theory from commutative fields to non-commutative fields. A different treatment of these problems is evolved in this note. The characteristic feature of the method employed is the (elementary) proof of general reduction theorems which permit the translation of the Galois theory of any class of groups of automorphisms of commutative fields into a theory concerning groups of automorphisms of more general domains like non-commutative fields and linear systems over commutative fields. This method permits therefore, in particular a generalization of Galois' theory of infinite algebraic extensions. (Received July 25, 1939.)

299. Reinhold Baer: *Nilpotent groups and their generalizations.*

The following three properties of groups are equivalent for finite groups and may be used as definitions of nilpotent groups: (1) The group is swept out by its ascending central-series. (2) The group is a direct product of p-groups. (3) If S and $T<S$ are subgroups such that there is no subgroup between T and S, then T is a normal subgroup of S. For infinite groups the situation changes and an investigation of the interrelations of these three properties is made. A seemingly only slightly stronger property than (3) is: (4) if S and $T<S$ are subgroups, and if there exists at most one subgroup between S and T, then T is a normal subgroup of S. It may be proved that a group which satisfies (4) and which satisfies either (1) or does not contain elements of infinite order has the property that all its subgroups are normal subgroups. (Received July 25, 1939.)

300. Miriam F. Becker and Saunders MacLane: *The minimum number of generators for inseparable algebraic extensions.*

An inseparable finite extension of a field K cannot be obtained, in general, by the adjunction of a single element to K. In this paper the minimum number of generators is determined for all extensions of a fixed base field and also for specific extensions. In applying the criterion for the minimum number of generators it is necessary to compute an invariant, the degree of imperfection, of the given field. The effect of algebraic extensions, finite and infinite, upon the degree of imperfection is thus included. (Received June 27, 1939.)

301. P. O. Bell: *Projective invariants of a curve on a surface.*

Let x denote Fubini's normal coordinates for a general point of a surface S_x in ordinary space. One of the uses to which the author has applied his concept of the R_λ-associate of a line is the geometric characterization of the projective arc-length of a finite arc of a curve on S_x. The elements obtained in this characterization serve to determine geometrically the sequences of points: d^ix/ds^i and $d^ix/ds^i \pm 2d^{i-1}x/ds^{i-1}$, $(i=1, 2, \cdots, n)$, where differentiation is with respect to arc-length s along a curve C_λ of S_x. These determinations form the groundwork for geometric definitions of a number of important invariants of C_λ at x. Definitions are given for the second order invariants known as projective curvature and asymptotic curvature of a curve C_λ at x and for the third order invariant known as projective torsion. Finally a method is given for the geometric characterization of invariants of any order. (Received July 31, 1939.)

302. Stefan Bergmann and W. T. Martin: *On the existence of a function of integrable square which satisfies an infinite system of integral relations.*

The paper investigates the conditions under which there exists a (complex) function $f(t_1, t_2) \varepsilon L^2$ for $0 \leqq t_k \leqq 1$, $k=1, 2$, which satisfies a system of equations $\int_0^1 \int_0^1 f(t_1, t_2) t_1^{\lambda_{1n}} t_2^{\lambda_{2n}} dt_1 dt_2 = X_n$, where $\{\lambda_{1n}, \lambda_{2n}\}$, $\mathfrak{R}\{\lambda_{kn}\} > -\frac{1}{2}$ and $\{X_n\}$ are given systems of complex numbers. It is shown that this problem possesses a solution if and only if there exists a function $F(z_1, z_2)$ of two complex variables regular in $\mathfrak{R}(z_k) > 0$, satisfying $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x_1+iy_1, x_2+iy_2)|^2 dx_1 dx_2 < c < \infty$ and taking on the values X_n at the points $[(\frac{1}{2}+\lambda_{1n})i, (\frac{1}{2}+\lambda_{2n})i]$. Since an infinite series can be given whose convergence is a necessary and sufficient condition for the existence of a solution of this interpolation problem (see Bergmann, Bulletin de l'Institut de Mathématiques et Mécanique à l'Université de Tomsk, vol. 1 (1935–1939), pp. 242–256, esp. p. 250), the solution is obtained explicitly when it exists. By using the theory of regions with distinguished boundary surfaces the authors treat analogous problems with other integral conditions. (Received July 28, 1939.)

303. Garrett Birkhoff: *Neutral elements in general lattices.*

An element a of a lattice is called neutral if every triple $\{a, x, y\}$ generates a distributive sublattice. It is proved that the neutral elements of any lattice L form a (distributive) sublattice, consisting of those elements carried into $[I, 0]$ under isomorphisms of L with sublattices of direct products. Further, complements of neutral elements, when they exist, are unique and neutral. The sublattice of complemented neutral elements may be called the "center" of a lattice: it consists of those elements carried into $[I, 0]$ under isomorphisms of the lattice with direct products. (Received July 17, 1939.)

304. Richard Brauer: *On groups whose order is divisible by the first power of a prime.* II. Preliminary report.

Using the results of an earlier paper (abstract 45-5-164), groups G are studied which satisfy the following conditions: (1) the order g of G is of the form $g=pg'$, where p is a prime and g' not divisible by p, (2) the elements of order p commute only with their own powers, (3) G is simple. The first two of these conditions are, for instance, satisfied for all transitive permutation groups of prime degree p, and (3) does not imply any essential restriction in this case. The order g is of the form

$g=qp(1+np)$ where $q|(p-1)$. It is shown that if $n<(3p+13)/4$, then either G is cyclic, or $G\cong LF(2, p)$, or p is a prime of the form $2^h\pm 1$ and $G\cong LF(2, 2^h)$, or $G\cong LF(3, 3)$. Incidentally, it can be shown that $LF(3, 3)$ is the only simple group of the order 5616. (Received July 28, 1939.)

305. R. S. Burington: *On transient similarity and equivalence in linear networks.*

Let $A_i\equiv(a^i_{rs})$, where the a^i_{rs} are real polynomials in λ, be a network matrix of order n for an n-mesh linear network N_i. N_i is said to be *transiently similar* to N_j if A_i and A_j have the same real Segre characteristic. In the present paper the concept of transient similarity is used to give a complete classification of the set of all n-mesh networks with respect to transient behavior. The case $n=2$ is considered in detail and gives a complete and improved solution of the problem undertaken by W. Quade (*Klass. der Schwing. in gekoppelten Stromkreisen*, Leipzig, 1933). The connections between steady-state equivalence, transient similarity and transient equivalence are considered in detail. (Received July 31, 1939.)

306. S. S. Cairns: *Homeomorphisms between topological manifolds and analytic manifolds.*

The theorems here stated are obtained from previously announced results (abstract 45-3-110) with the aid of methods due to Hassler Whitney (*Differentiable manifolds*, Annals of Mathematics, (2), vol. 37 (1936), pp. 645–680). A polyhedral manifold P^r in a euclidean n-space E^n is said to be *in normal position* if there exists a continuously varying $(n-r)$-plane $\pi^{n-r}(p)$ transversal to P^r at p (cf. abstract 45-3-110). Theorem I: It is possible, by introducing suitable coordinate systems to make a given topological manifold M analytic, with an analytic Riemannian metric, if and only if M is homeomorphic to some polyhedral manifold in normal position. Since every P^3 can be put in normal position, the following theorem results. Theorem II: Any topological 3-manifold which can be triangulated can be made analytic, with an analytic Riemannian metric. (Received July 31, 1939.)

307. H. H. Campaigne: *Suschkewitsch correspondences.*

Suschkewitsch (Transactions of this Society, vol. 31 (1929), pp. 204–214) has considered quasi-groups G^* obtainable from a group G by making a permutation in the headline of the Cayley multiplication table. This suggests the triple correspondence of G^* to G, where each element g^* of G^* determines three elements g_1, g_2, g_3 of G, and each element of G is determined by three of G^*, such that $a^*b^*=c^*$ if and only if $a_1b_2=c_3$. Suschkewitsch studied such a correspondence for the special case $g_1=g_3$. These correspondences form an equivalence relation among groupoids. The groupoid G^* is a quasi-group if and only if G is a quasi-group. A second correspondence, $G^*\otimes G$, is also studied. This is a one-to-one correspondence $g^*\rightarrow g$, such that $ab=c$ if and only if $a^*c^*=b^*$. The lattice of all subgroupoids of G^* is isomorphic to that of G. The group of automorphisms of G^* is isomorphic to that of G. The associative law is not preserved under either of these correspondences. (Received August 2, 1939.)

308. H. S. M. Coxeter: *A new form of coordinates for the polytope 2_{21} whose twenty-seven vertices correspond to the lines on the general cubic surface.*

The new coordinates proposed for the 27 points are the real and imaginary parts of the complex numbers $(0, \omega^\lambda, -\omega^\mu)$, $(-\omega^\mu, 0, \omega^\lambda)$, $(\omega^\lambda, -\omega^\mu, 0)$, where ω^λ, $(\lambda=0, 1, 2)$, are the cube roots of unity. These complex numbers are the first three coefficients in the equations of the 27 linear complexes "of the first kind" which are permuted among themselves by the quaternary collineation group of order 25920 (Burkhardt, Mathematische Annalen, vol. 41 (1892), p. 323). The remaining three coefficients are their respective conjugates. When 0, 1, ω, ω^2 are interpreted as the marks of the field $GF[2^2]$, the collineation group becomes $HO(4, 2^2)$, and the linear complexes become special, so that they can be replaced by single lines of the finite geometry $PG(3, 4)$ (Frame, this Bulletin, vol. 44 (1938), p. 659). These are, in fact, the 27 lines on the "cubic surface" whose 45 points satisfy the congruence $x_0^3+x_1^3+x_2^3+x_3^3\equiv 0$ in the Galois field. The remaining 40 points of the finite geometry are found to correspond to 40 concurrent planes of euclidean eight-space which together contain the 240 vertices of the polytope 4_{21}. (Received July 25, 1939.)

309. Bernard Dimsdale: *Degree of approximation by linear combinations of powers.*

Szász (Mathematische Annalen, vol. 77 (1916), pp. 478–493) has shown that any continuous function may be uniformly approximated by a linear combination of powers for which the exponents satisfy certain conditions. The present paper, based on a part of the work of Szász, contains a corresponding discussion of degree of approximation. With certain further restrictions on the sequence of exponents, results are obtained which are comparable in form and generality with those that hold for approximation by ordinary polynomials in successive integral powers of the variable. (Received July 31, 1939.)

310. J. E. Eaton: *Theory of cogroups.*

A cogroup is a special multigroup in which every submultigroup is reversible. Since the coset decomposition of any group with respect to a subgroup is a cogroup, a few of the results are of group theoretical interest. (Received July 24, 1939.)

311. Benjamin Epstein: *Growth properties of analytic functions of two complex variables.*

Using the theory of domains with distinguished boundary surfaces (ausgezeichnete Randflächen) recently established by S. Bergmann, the author obtains theorems concerning the growth in certain non-analytic three-dimensional hypersurfaces $\mathfrak{S}^3$ of analytic functions $f(z_1, z_2)$, $z_k=x_k+iy_k$, $k=1, 2$, regular in $\mathfrak{M}^4$: $x_k\geqq 0$. For instance, if $|f(iy_1, iy_2)|$ is bounded for $-\infty\leqq y_k\leqq+\infty$ and if $|f(iy_k, z_k)|\leqq C|r_k|^{1-\epsilon}+|y_k|^{\alpha_k}$ (where α_k can be explicitly found), then either $|f(z_1, z_2)|\leqq 1$ in $\mathfrak{M}^4$ or max $|f(z_1, z_2)|$ must exceed a certain lower limit, in every $\mathfrak{S}^3$, for its growth. Furthermore theorems are obtained about zero-lines in these hypersurfaces $\mathfrak{S}^3$. Analogous results are obtained for the case in which $\mathfrak{M}^4$ is $|z_1|<\infty$, $-\pi/2k_1\leqq\arg z_1\leqq\pi/2k_1$, $|z_2|<\infty$, $-\pi/2k_2\leqq\arg z_2\leqq\pi/2k_2$. (Received July 26, 1939.)

312. F. A. Ficken: *Cones and vector spaces.*

This paper is concerned with the relation of an arbitrary vector space V to a quadric cone C in a real vector or centered affine n-space. By exploiting the contact-space of a V tangent to C, and using the most elementary concepts of lattice theory, a simple, complete, and highly intuitive geometric account is given of the intersection of V with C. The argument yields many known results, and leads to a classification of vector spaces which is shown to be the equivalence-classification under the Lorentz group of transformations which leave C fixed. (Received July 31, 1939.)

313. Bernard Friedman: *Fourier coefficients of bounded functions.*

Let $f(x)$ be a bounded measurable function in $(-\pi, \pi)$ such that $|f(x)| \leqq 1$. Let a_n, b_n be the Fourier coefficients of $f(x)$ so that $\pi a_n = \int_{-\pi}^{\pi} f(x) \cos nx\, dx$, $\pi b_n = \int_{-\pi}^{\pi} f(x) \sin nx\, dx$; $n = 1, 2, \cdots$. Obviously $|a_n|$, $|b_n| \leqq 4/\pi$. This paper shows that these coefficients must lie in definite regions. For example, b_1 and b_2 must lie in the following region: $-16 + \pi^2 b_1^2 \leqq 4\pi b_2 \leqq 16 - \pi^2 b_1^2$. The above inequalities are the best possible in the sense that if a set of a, b make the inequality an equality, there exists a bounded function having those numbers as its Fourier coefficients. Also, for any bounded $f(x)$ and any integer n, there exists a function $g_n(x)$ taking on only the values $+1$ and -1 such that the first n Fourier coefficients of $f(x)$ and $g_n(x)$ are the same. Incidently there is proved the following result: If $F(x)$ is a bounded measurable function such that $|F(x)| \leqq \frac{1}{2}$ then $\int_0^{\pi} F(x) \cos x dx \leqq \cos (\int_0^{\pi} F(x) dx)$. (Received August 2, 1939.)

314. B. E. Gatewood: *Thermal stresses in long cylindrical bodies.*

The problem of finding thermal stresses and displacements in long cylinders is reduced to obtaining a particular integral for the Poisson equation $\nabla^2 V = kT$, where T is the temperature distribution and k is a constant, and to solving the biharmonic equation $\nabla^4 U = 0$ when the first derivatives of the function U are known on the boundary of the crosssection of the body. These two differential equations are related by the condition that on the boundary the first derivatives of U are given in terms of the first derivatives of V. The biharmonic equation is solved by use of two analytic functions (Muschelisvili, Bulletin de l'Académie des Sciences de Russie, vol. 13 (1919); Mathematische Annalen, vol. 107 (1933)). The region within which the solution is desired is mapped conformally on the unit circle by a rational function, and from the boundary values on the unit circle the required analytic functions are determined directly by the use of the Cauchy integral formula. The methods are extended to multi-connected bodies and to bodies composed of several different materials. Special problems of the circular cylinder, the hollow circular cylinder, the concentric and eccentric composite circular cylinders are solved. (Received July 24, 1939.)

315. Michael Goldberg: *New five-bar and six-bar linkages in three dimensions.*

A closed chain of n hinged links in three dimensions has, in general, $n-6$ degrees of freedom. When n is less than 7 the chain is generally rigid; a movable chain is said to be "paradoxical." Bricard (*Leçons de Cinématique*, Paris, 1927) lists all the known paradoxical chains. His list contains chains of four links and of six links, but it does not contain any chains of five links. This paper describes the structure and motion of a new family of chain linkages of five links, and also, two new types of six links. (Received July 21, 1939.)

316. L. M. Graves: *On the fundamental lemma of Haar in the calculus of variations.*

A "classical form" of fundamental lemma for double integrals may be stated as follows: (a) let the functions $U(x, y)$, $V(x, y)$, $W(x, y)$, $\partial U/\partial x$, $\partial V/\partial y$ be continuous on a bounded open set A; and (b) let $\iint_A [U\xi_x + V\xi_y + W\xi] dx\, dy = 0$ for every ξ of class C' vanishing on the boundary B of A; then (c) $W=\partial U/\partial x+\partial V/\partial y$ on A. Let $U_\epsilon(x, y)$, $V_\epsilon(x, y)$, $W_\epsilon(x, y)$ be the integral means of U, V, W respectively, taken over the square $S(x, y, \epsilon)$ with center (x, y) and side 2ϵ. Then when the hypothesis (a) is replaced by: (a_1) let the functions U and V be bounded and measurable and W be integrable over A; the conclusion (c) may be replaced by: (c_1) for every (x, y, ϵ) such that $S(x, y, \epsilon)$ is in A, the partial derivatives $\partial U_\epsilon/\partial x$ and $\partial V_\epsilon/\partial y$ exist and $W_\epsilon=\partial U_\epsilon/\partial x+\partial V_\epsilon/\partial x$. Previous proofs of a related form of the Haar fundamental lemma under the weak hypothesis (a_1) have obtained an equation holding only for almost all rectangles. The author also shows that when (a_1) is replaced by the somewhat stronger: (a_2) let the functions U and V be continuous on A and let W be integrable over A; then (c_1) may be obtained very simply as a consequence of the "classical form" of the fundamental lemma stated above. (Received July 27, 1939.)

317. L. M. Graves: *Some general approximation theorems.*

In this paper comparatively simple proofs are given for some general theorems on the approximation of a given function by "smoother" functions. Well known theorems on approximation by polynomials, by trigonometric sums, and by integral means are special cases. The proofs are formulated for functions of any finite number of variables. Theorems on the term-by-term differentiation and integration of the approximating sequence are included. When the given function is continuous, and vanishes on a closed point set E, it is shown that the approximating functions may be required to vanish on a neighborhood of E while possessing as many continuous derivatives as may be desired. (Received July 22, 1939.)

318. D. W. Hall (National Research Fellow): *On pointwise periodic homeomorphisms in the plane.*

Let M be a continuous curve on the 2-sphere S_2 which is not separated by the omission of any pair of points. Let $T(M)=M$ be pointwise periodic. Then T is periodic and may be extended to a periodic transformation on S_2 and hence must be identical with an elementary euclidean transformation. As a result the negative solution to a problem of W. L. Ayres (see abstract 45-1-87) is obtained. (Received July 28, 1939.)

319. D. W. Hall (National Research Fellow) and J. L. Kelley: *On continuous collections of orbits.*

Let M be a compact metric space and $T(M)\subset M$ be a homeomorphism. In order that the collection of point orbits in M under T form a continuous collection, each of the following conditions is necessary and sufficient: (1) if K is a closed set in M, then K together with all its images forms a closed set; (2) T is pointwise periodic and almost periodic, that is, every point orbit is finite and for $\epsilon>0$ there exists a positive integer L, such that in every set of L consecutive integers there is an n such that $\rho[x, T^n(x)]<\epsilon$ for all $x \in M$. (Received July 28, 1939.)

320. P. R. Halmos: *On a necessary condition for the strong law of large numbers.*

Let $\{x_n\}$ be a sequence of independent chance variables, and $\bar{x}_n=x_n$ when $|x_n|\leqq n$, $\bar{x}_n=0$ otherwise. Write $b_n=E[\bar{x}_n-E(\bar{x}_n)]^2$, and suppose that $1/n\sum_1^n E(\bar{x}_i)\to 0$. The following theorem (known as the strong law of large numbers) is an easy consequence of a result due to Kolmogoroff: If $x_n/n\to 0$ with probability 1 and if $\sum_1^\infty b_n/n^2<\infty$, then $1/n\sum_1^n x_j \to 0$ with probability 1. The converse of this theorem is not known and it is the purpose of this paper to make a contribution in this direction by proving the following theorem: If $1/n\sum_1^n x_j \to 0$ with probability 1, then $x_n/n \to 0$ with probability 1, and $\sum_1^\infty b_n/n^{2+\epsilon} < \infty$ for all $\epsilon > 0$. (Received July 20, 1939.)

321. O. G. Harrold: *Topological properties of rectifiable continua.*

It is shown in this paper that the property of a compact continuum of being a topological image of a rectifiable continuous curve is cyclicly extensible and reducible. If M is a compact continuum of finite degree (Kamiya), there exists a topological image M^1 of M and a continuous transformation T of the unit interval I such that $T(I) = M^1$ and the length of M^1 with respect to T is finite. If M is any rectifiable continuous curve, M is of finite degree. (Received August 3, 1939.)

322. Philip Hartman and R. B. Kershner: *On some limit theorems for number theoretic functions.*

In the study of number theoretic functions various authors have proved a number of limit results of which the following is typical: lim inf $\phi(n)$ log log $n/n = e^{-\gamma}$ where $\phi(n)$ is the Euler ϕ-function and γ is the Euler-Mascheroni constant. In this paper several general theorems of this type are proved by very elementary methods. These general theorems include all of the classical results as special cases. (Received August 1, 1939.)

323. Olaf Helmer: *The principal ideal theorem in domains of integral functions.*

Let $K\langle z\rangle$ denote the domain of integral functions $f(z) = \sum a_n z^n$ with coefficients a_n in K, K being an arbitrary number field. P may stand for the field of rational numbers. Every $K\langle z\rangle$ is an integral domain, and one can speak of the ideals in $K\langle z\rangle$. This paper is concerned with the question as to whether there are any but principal ideals in $K\langle z\rangle$. The answer depends upon the precise definition of "ideal" that is employed. In contradistinction to the classical definition, Prüfer (Journal für die reine und angewandte Mathematik, vol. 168 (1932)) introduced a narrower ideal concept which comprises only the ideals (in the classical sense) with a finite basis. And the answer can now be formulated as follows. A: Every $K\langle z\rangle$ contains ideals in the classical sense which are not principal ideals. B: If only ideals in the Prüfer sense are admitted, then every ideal in $K\langle z\rangle$ is a principal ideal. (Received July 28, 1939.)

324. T. R. Hollcroft: *Anomalous systems of primals.*

An anomalous system of primals is a system for which the invariant effective dimension is greater than the invariant virtual dimension. In 1929, B. Segre discovered an anomalous (*irregolare*) system of plane curves with dependent cuspidal invariants. In the present paper it is shown that anomalous systems of primals exist with various types of distinct double points. In particular, the system $\sum f_i^2 = 0$ ($i = 1, \cdots, r$, each f_i being a nonsingular primal of order α in S_r) has α^r nodes lying at the intersections of the ∞^{r-1} linear system $\sum \lambda_i f_i = 0$. A general expression for the anomaly of this system is found. For $r = 3$, this is a system of surfaces of order α with α^3 nodes lying at the intersections of a net of surfaces of order α. The anomaly of this system is $(\alpha-1)(\alpha-2)(\alpha-3)/6$. (Received July 27, 1939.)

325. D. H. Hyers: *A generalization of Fréchet's differential.*

Several definitions of differentials in linear topological spaces have already been given. (See A. D. Michal and E. W. Paxson, Comptes Rendus, Warsaw Academy of Sciences, vol. 29 (1936), pp. 106–121, and A. D. Michal, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 340–342.) In the present note the differential of Fréchet is generalized by making use of pseudo-norms in place of norms. The definition of this "F-differential" applies to functions with arguments in any linear topological space and values in any other linear topological space. The usual rule for the differentiation of iterated functions is shown to hold. Every F-differentiable function is M-differentiable in the sense of Michal. When the linear topological spaces can be normed, the F-differential coincides with the Fréchet differential. It is still an open question whether this is also true of the M-differential. (Received July 25, 1939.)

326. Dunham Jackson: *Orthogonal polynomials with auxiliary conditions.*

In recent papers before section meetings of the Association the writer has presented examples of systems of orthogonal polynomials subject to linear homogeneous auxiliary conditions. The conclusions noted in those papers for the special cases considered there are now extended to form a theory of some degree of generality, at least on the formal side, while the theory of convergence also is carried somewhat beyond the stage previously reached. (Received July 31, 1939.)

327. Fritz John: *Special solutions of certain difference equations.*

It was proved by H. Bohr, that any continuous, convex solution of the difference equation $y(x+1)-y(x)=\log x$ differs from $\log \Gamma(x)$ by a constant. Recently A. Meyer proved (Acta Mathematica, vol. 70, p. 59), that the functional equation $y(x+1)=1/x\cdot y(x)$ has only one convex solution. In this paper the author proves theorems of this type for more general difference equations, for example: Every two continuous, convex solutions of the difference equation $\sum_{i=0}^{n} a_i f(x+i)=g(x)$ with constant coefficients differ at most by a constant, if (1) all roots of the characteristic equation are simple and of absolute value 1, (2) $g(x)$ is continuously differentiable, (3) either $\lim_{x\to\infty} g'(x)=0$, or $\sum_i a_i=0$ and $\lim_{x\to\infty} g'(x)/x=0$. (Received July 17, 1939.)

328. Wilfred Kaplan: *Regular curve-families filling the plane.*

Let a family F of curves in the plane be given such that (a) each curve is the homeomorphic image of a circle or of an open interval, (b) through each point of the plane passes one and only one curve, (c) locally the family is homeomorphic with a set of parallel lines. A family of this type can be obtained as the level curve family of a continuous function $f(x, y)$. This paper establishes the converse: every such family F is the level-curve family of a non-trivial continuous function $f(x, y)$. This is proved by means of a "normal" subdivision of the family F into nonoverlapping pieces, in each of which the family is homeomorphic with the parallel lines of a half-plane. (Received July 28, 1939.)

329. Edward Kasner and J. J. De Cicco: *Trihornometry of second order.*

This paper is a continuation of the paper by the authors, *Conformal geometry of horn angles of second order* (Proceedings of the National Academy of Sciences, vol. 24

(1939), pp. 393–400). The laws connecting the conformal invariants of a trihorn of second order are obtained. This configuration consists of three curves (C_1, C_2, C_3) which pass through a common point in a common direction such that all three curves have the same curvature but no two of them have the same rate of variation of curvature at the common point. A trihorn possesses nine conformal invariants: the three sides or measures M_{12}, M_{23}, M_{31}, the three direct dihorn angles α_{12}, α_{23}, α_{31}, and the three reverse dihorn angles α_{21}, α_{32}, α_{13}. The laws connecting these nine parts are analogous to the angular identity, and the laws of sines and cosines of ordinary plane geometry. In general, four parts of a trihorn have to be given in order to determine the remaining five parts. (Received July 20, 1939.)

330. W. S. Kimball: *Partial derivatives of derivatives.*

The parallel displacement method used previously with line integrals and the calculus of variations is here extended to include second and higher partial derivatives where the slope y' appears in the vector integrand. The need for partial derivatives of the slope requires a new definition of the derivative admitting parallel displacement differentiation: The derivative of y with respect to x is the quotient of $dy_1 = Ln$ by $dx = \Delta x = n$, where n represents unit positive magnitude, and L is the limit heretofore defined as the derivative. The new derivative equals the same limit as formerly and involves no new departure except that it particularizes the differentials whereby dx for the independent variable must be unit positive magnitude n, and with corresponding restriction on dy. The rate concept using parallel displacements leads uniquely to the new partial derivatives, which are functions of units in contrast with classical derivatives. These partial derivatives of y' are required by mathematical consistency to fill a gap left by classical differentiation when length elements as well as density vary along a wire. (Received July 31, 1939.)

331. M. S. Knebelman: *Scalar invariants of the vector vortex excess.*

In a recent paper (American Journal of Mathematics, vol. 61, pp. 461–472), Hermann Weyl introduces the scalar invariants H_e defined over a Riemann space C_ν embedded in a spherical space of curvature $1/\nu^2$ where e is even and $0 \leqq e \leqq \nu$. In this note it is shown that, except for a numerical factor, H_e is the trace of the $e/2$th compound of the matrix $H_{\alpha\beta}^{\lambda\mu}$, where the pairs of different indices are arranged in a certain order, and that the invariants H_i for $i = 1, 2, \cdots, \nu(\nu-1)/2$ form a complete set of scalar invariants. (Received July 3, 1939.)

332. R. E. Langer: *On the stability of the laminar flow of a viscous fluid.*

The effect of the disturbance in the form of an elementary wave of length $2\pi/\alpha$ upon the flow of a fluid between parallel plane boundaries has been variously studied by other authors. With customary assumptions of a physical nature the problem may be made to depend upon the solution of an ordinary differential equation of the fourth order subject to four boundary conditions. The coefficients of this equation involve the wave length of the disturbance, the Reynold's number of the fluid, R, and a parameter C. For characteristic values of the latter the problem is solvable. Depending on whether the imaginary part of any characteristic value is positive or negative, the associated motion of the fluid is stable or unstable. This paper discusses the asymptotic nature of the configuration when αRC is large. It is shown that all characteristic values lie in the strip of the complex plane $0 < \Re(C) < 1$, $0 < \Im(C)$ augmented by a small neighborhood of the real interval from 0 to 1. Thus, except possibly for some particular motions near the borderline of turbulence, other motions of the fluid are shown to be stable. (Received July 31, 1939.)

333. C. G. Latimer: *A class of diophantine equations.*

Let α, β be rational integers such that (a) one is negative, (b) not both are in the form $4n+3$ and (c) $\alpha\cdot\beta$ is not divisible by a square greater than 1. $f\equiv x^2+\alpha y^2+\beta z^2+\alpha\beta w^2$ is the norm of the general element in a ring $\mathfrak{G}$ of integral elements in a quaternion algebra $\mathfrak{A}$. If d is the fundamental number of $\mathfrak{A}$ it is known that $\Delta\equiv 2\alpha\cdot\beta/d$ is an integer. Let $\Delta=2^\epsilon\cdot\delta$ where $\epsilon=0$ or 1 according as Δ is odd or even. In this paper we determine all solutions in integers of $f=u_1\cdot u_2\cdot\cdots\cdot u_n$, where the u's are prime to δ. This result is obtained after showing that every one-sided ideal in $\mathfrak{G}$ which contains a rational integer prime to δ is a principal ideal. For the case where $\alpha=1$, $\beta=-3$ our result is the same as that obtained by Dickson (*Algebren und ihre Zahlentheorie*, p. 173, Theorem 13). Similar results may be obtained for the equation $f_1=u_1\cdot u_2\cdot\cdots\cdot u_n$ where f_1 is any form which may be transformed into f by a transformation with integral coefficients. (Received July 31, 1939.)

334. D. H. Lehmer: *Certain congruential properties of spherical harmonics.*

Let $P_n(x)$ be Legendre's polynomial and let $Q_n(x)$ be the second solution of his differential equation. Finally let $Z_n(x)$ denote the familiar polynomial of degree $n-1$ defined by $P_n(x)[\log(x+1)-\log(x-1)]-2Q_n(x)$, frequently designated by $2W_{n-1}(x)$. Several identical congruences connecting P, Q, and Z are derived in this note of which the following are typical: (1) $Z_{p-1}(x)\equiv 0 \pmod p$ identically in x; (2) $pZ_{p+r}(x)\equiv 2P_r(x) \pmod p$. In both cases p is a prime. (Received July 25, 1939.)

335. Jakob Levitski: *On multiplicative systems.*

In the present note it is proved that each nil-ring, in a ring which satisfies a certain maximum condition, is nilpotent. This is an extension of a result first found by Charles Hopkins (Duke Mathematical Journal, vol. 4 (1938), pp. 664–667) concerning rings which satisfy a certain minimum condition. In particular follows the theorem: Each right or left nil-ideal of a ring which satisfies the minimum or the maximum condition for the right ideals, is nilpotent. This solves a problem proposed by Koethe. The results obtained are based upon two general theorems, which are possibly of independent interest, on multiplicative nil-systems. (Received July 21, 1939.)

336. D. C. Lewis: *The separability of Pfaffian transformations.*

A nonsingular analytic transformation T of a region of n dimensional euclidean space into another such region is called Pfaffian with respect to a linear differential form ω, if it admits $\int\omega$ as a relatively invariant line integral. If coordinates $x_1,\cdots,x_n$ are introduced in such a manner that ω can be written in the form ω' plus an exact differential, where ω' depends only on $x_1,\cdots,x_{2k}$, $(2k<n)$, then T takes the form: $\bar{x}_i=f_i(x_1,\cdots,x_{2k})$, $i=1,\cdots,2k$; $\bar{x}_j=g_j(x_1,\cdots,x_n)$, $j=2k+1,\cdots,n$; provided that ω' cannot be expressed in fewer than $2k$ variables. This result is closely analogous to a classical result of Lie and Koenig for differential equations. Applications are made to Pfaffian transformations admitting invariant manifolds. (Received July 17, 1939.)

337. Ingo Maddaus: *On types of "weak" convergence in normed vector spaces.* Preliminary report.

This paper is concerned with the study of weak convergence and convergence which is weaker than weak convergence. The fundamental notions, with postulational restrictions, are (1) an association (x, y) to any pair x and y of elements of the space, which is of the nature of a generalized "metric," the range not being restricted to real numbers, (2) a real number or norm corresponding to every finite number of elements and to every finite number of associations of (1). A limit notion, $x_n \to_H x$, is defined by means of the norm of a finite number of associations of (1). In the space of all continuous functions on $0 \leqq t \leqq 1$, $x_n \to_H x$ is equivalent to $x_n(t) \to x(t)$ for each t. Realizations of $x_n \to_H x$ are given for (l^p), (L^p), $(p \geqq 1)$, and for (c). By different interpretations of (1) and (2), $x_n \to_H x$ is found to be equivalent to weak convergence and to a still weaker convergence in (l^p), (L^p), $(p \geqq 1)$. Properties of linear, H-continuous functionals and transformations are studied and general forms are obtained. (Received July 31, 1939.)

338. I. Malkin: *On the elastic problem of the plate.* Preliminary report.

A solution of the well-known Michell type of the elastic problem of the plate is obtained explicitly by using an operational expansion of the biharmonic function of the space in the form $u = u_0 + \sum(-1)^{n+1}(z^{2n}/(2n)!)D^{n-1}(nU_0 - Du_0) + (z/1!)u_1 + \sum(-1)^{n+1}(z^{2n+1}/(2n+1)!)D^{n-1}(nU_1 - Du_1)$ wherein u_0, u_1, U_0, U_1 are functions of x and y, and D is the plane Laplacian operator, the displacement components being biharmonic functions in the case of vanishing body forces. With the aid of such expansions for the displacement components the solution of the elastic problem of the plate is expressed in terms of six arbitrary functions of x and y, which suffice to satisfy the boundary conditions of the plane surfaces of the plate, while on the cylindrical surface they are fulfilled in using the principle of de St. Venant. In the case of a plate under normal load the problem is reduced to determination of two arbitrary functions. This case is worked out in detail in the paper and illustrated by a numerical example in finite polynomials. In connection with previous publications on analogous solutions of the harmonic problem (conduction of heat), the present paper is intended as a basis for the treatment of the thermoelastic problem of plates and shells. (Received July 28, 1939.)

339. N. H. McCoy: *A generalization of Ostrowski's theorem on matric identities.*

Recently A. Ostrowski (Quarterly Journal of Mathematics, vol. 9 (1938), pp. 241–245) has proved a theorem which is, in a certain sense, an analogue for several matrices of Frobenius' theorem on the minimum equation of a matrix. In the present note, one part of Ostrowski's theorem is generalized to the case in which the elements of the matrices are elements of an arbitrary commutative ring R with unit element. By making one additional restriction on R, the other part of the theorem is similarly generalized. (Received August 4, 1939.)

340. E. J. McShane: *An estimate for the Weierstrass $\mathcal{E}$-function.*

It is known that if a curve C: $y = y(x)$, $a \leqq x \leqq b$, satisfies the differential equations of a Bolza problem and is nonsingular, and along it the strengthened

Weierstrass condition II'_N holds, then for all (x, y) near C an inequality of the type $\mathcal{E}(x, y, y', Y', \lambda) \geqq \delta(\epsilon)\|y'-Y'\|$, $(\delta(\epsilon)>0)$, holds whenever (x, y, y') and (x, y', Y') are admissible and $\|y'-Y'\| \geqq \epsilon$. Here it is shown that the hypothesis of nonsingularity can be discarded. Incidentally, it is proved that even for singular curves the transformation to canonical variables is topological, provided that the strengthened Weierstrass condition is satisfied. (Received July 28, 1939.)

341. E. J. McShane: *A remark concerning sufficiency theorems for the problem of Bolza.*

By virtue of recent theorems of Hestenes, Morse and Reid, a curve $y=y(x)$ is known to be a minimizing curve for a Bolza problem if there are multipliers $\lambda_0=1$, $\lambda_\alpha(x)$ with which it is an extremal and satisfies the strengthened Weierstrass, Clebsch and Jacobi conditions. Proofs of necessary conditions yield only that there are multipliers $\lambda_0 \geqq 0$, $\lambda_\alpha(x)$ with which the Euler, transversality, Weierstrass and Clebsch conditions hold. Here it is shown that the discrepancy concerning λ_0 is easily removed; the sufficiency theorem remains valid if the hypothesis $\lambda_0=1$ is replaced by $\lambda_0 \geqq 0$. (Received July 28, 1939.)

342. E. J. McShane: *Generalized curves.*

A generalized curve in the sense of L. C. Young is a system consisting of a set of functions $y=y(t)$, $a \leqq t \leqq b$ (where y is a vector in ν-space), and a linear average $\mathcal{M}[t; \Phi(r)]$ defined on the space of continuous functions $\Phi(r)$ (r also is a vector in ν-space); there are of course certain restrictions on the $y(t)$ and the $\mathcal{M}$. Ordinary curves are special cases obtained by choosing $\mathcal{M}[t; \Phi(r)] = \Phi(y'(t))$. In the present paper Young's theory of these curves is presented and extended. The space of generalized curves can be so metrized that every integral of the calculus of variations is continuous in the metric, and every bounded closed set in the space of curves is compact. This leads at once to existence theorems concerning the existence of minimizing generalized curves for generalized-curve problems of Bolza type. (Received July 28, 1939.)

343. E. J. McShane: *Necessary conditions for a minimum in generalized-curve problems of the calculus of variations.*

The problem studied is that of minimizing a functional (integral plus function of end points) in the class of generalized curves (see preceding abstract) which satisfy certain differential equations and end conditions. Only parametric problems are considered. Conditions are established which are necessary in order that a generalized curve afford a strong relative minimum for the problem. Analogues of the multiplier rule, transversality conditions, Weierstrass condition, Clebsch condition and Dresden corner condition are established. (Received July 28, 1939.)

344. E. J. McShane: *Existence theorems for Bolza problems in the calculus of variations.*

For generalized-curve problems of Bolza it is easy to establish existence theorems (see preceding abstract 44-9-342). By use of the necessary conditions satisfied by such a minimizing curve, it can be shown that under suitable assumptions on the data of the problem this minimizing generalized curve is actually a curve in the usual sense. The theorems thus obtained are of considerable generality, and include at least interesting subcases of all existence theorems for single-integral problems known to the author. (Received July 28, 1939.)

345. A. D. Michal: *A differential in linear topological spaces.*

A new differential, a slight variant of the M-differential, is defined for functions with arguments and values in linear topological spaces. It is equivalent to the Fréchet differential whenever the linear topological spaces are Banach spaces. (Received August 1, 1939.)

346. A. D. Michal: *Fréchet differentials and M-differentials in Banach spaces.*

This paper characterizes a Fréchet differential in Banach spaces as a special linear Gateaux differential. If the linear topological spaces are specialized to Banach spaces, then the above result is used to characterize a Fréchet differential as a special M-differential. (Received August 1, 1939.)

347. A. D. Michal, R. Davis, and Max Wyman: *Polygenic functions in general analysis.*

Analytic functions in general analysis have been studied by R. S. Martin, Michal, and more recently by A. E. Taylor and others. In this paper, the authors begin the study of polygenic functions in general analysis. The directional Gateaux differential of a polygenic function is characterized for a certain large class of polygenic functions. (Received August 1, 1939.)

348. Deane Montgomery and Leo Zippin: *Topological transformation groups.*

This paper studies the action of a compact group G of homeomorphisms of a metrizable space E. The points of E which are on orbits of dimension greater than or equal to k form an open set. The local structure of an orbit of finite dimension is similar to the local structure of a topological group of the same dimension. If G acts effectively (only the identity leaving every point fixed) on a finite dimensional orbit, then G is itself finite dimensional; if G acts effectively in a locally euclidean space, G is finite dimensional. In this case, furthermore, if all orbits are locally connected, the group G is a Lie group. (Received July 17, 1939.)

349. Deane Montgomery and Leo Zippin: *Transformation groups characterizing certain three-manifolds.*

This paper deals with a space E on which there acts a compact connected transformation group G. The space E is subjected to various restrictions, chiefly that it be locally connected, separated by certain two-dimensional manifolds (if it contains them) and "barred off" locally into precisely two sides by any two-cell it may contain. Assuming further that G has at least one two-dimensional orbit, the authors show that the space E must be one of a very few simple types. In particular it is locally three-euclidean excepting, possibly, at the points of at most two orbits. The group G must be a Lie group: in fact, one of three. The paper is preliminary to a topological group characterization of three-space and its "rigid motions" paralleling Hilbert's work in the plane. Incidentally it is shown that the only two manifolds which can be coset spaces of a compact group are the sphere, the projective plane and the torus. (Received July 26, 1939.)

350. C. B. Morrey: *The problem of Plateau on a Riemannian manifold*. Preliminary report.

Let $\mathfrak{M}$ be a Riemannian manifold of class C'. It is assumed further that each point P of $\mathfrak{M}$ is interior to a neighborhood which possesses a map of class C' on the unit cube $R: |x_i| < 1$ in which P corresponds to the origin and such that the $g_{ij}(x_1, \cdots, x_N)$ are continuous on R and satisfy the condition $m \cdot \sum_{i=1}^N \xi_i^2 \leqq \sum_{i,j=1}^N g_{ij}(x_1, \cdots, x_N)\xi_i\xi_j \leqq M \cdot \sum_{i=1}^N \xi_i^2$ for each $(x_1, \cdots, x_N)$ in R and each $(\xi_1, \cdots, \xi_N)$, m and M being independent of P, $0 < m < M$. Let C be a simple closed curve bounding a surface of finite area in $\mathfrak{M}$. Then there exists a surface S in $\mathfrak{M}$ of least area bounded by C which possesses a (continuous) generalized conformal map on the closed unit circle $\bar{\Sigma}$. If $\mathfrak{M}$ is of class $C_\alpha^{(n+1)}$ (that is, if the $(n+1)$st derivatives in the transformations between overlapping coordinate systems satisfy Hölder conditions with exponent α) and if the g_{ij} are of class $C_\alpha{}^{(n)}$, then the functions occurring in a generalized conformal map of S are of class $C_\alpha{}^{(n)}$ on the interior Σ of $\bar{\Sigma}$; if $\mathfrak{M}$ and the g_{ij} are analytic, these functions are analytic. (Received July 28, 1939.)

351. David Moskovitz and L. L. Dines: *On the supporting-plane property of a convex body*.

This paper extends the results of an earlier paper (abstract 44-9-361) wherein the authors considered certain questions related to the notion of *convexity* in a linear space in which a metric and the notion of a hyperplane were introduced by means of a postulated inner product. The principal result of this paper is that in such a space a set which is closed, linearly connected, and which possesses inner points is completely supported at its boundary. This result together with results obtained in the earlier paper enable the authors to write the symmetrical theorem: If a subset of elements in the space considered is closed and possesses inner points, then the properties of being (i) *linearly connected* and (ii) *completely supported* at the boundary are logically equivalent. As in the earlier paper, the space is assumed to be complete but not assumed to be separable. (Received July 24, 1939.)

352. H. T. Muhly: *Valuations and infinitely near algebraic loci*. II. Preliminary report.

In this paper the results announced by the author in abstract 45-5-216 are extended to arbitrary algebraic varieties. The base condition imposed by a simple subvariety V_{r-2} of a V_r and by varieties $V_{r-2}^{(\alpha)}$ "infinitely near" V_{r-2} are interpreted arithmetically and studied from this point of view. Any valuation B of the field Σ of rational functions on V_r which has origin at V_{r-2} defines a succession $\{V_{r-2}^{(\alpha)}\}$ of varieties infinitely near to and in the successive neighborhoods of the proper variety V_{r-2}. The author studies a class of birational transformations of V_r which resolve the varieties of the various neighborhoods of V_{r-2} into proper varieties. These transformations are a natural generalization of the quadratic plane transformations and the monoidal space transformations. The results obtained respond to the geometric expectation which leads to their formulation, and they constitute a generalization of the classical theory of infinitely near points. (Received July 31, 1939.)

353. Albert Neuhaus: *On products of normal semi-fields*.

Let S of order n over K be a direct sum of t equivalent normal fields over K, so that S has a group of automorphisms G of order n which is transitive with respect to the t normal fields. Define a normal system (S, G) of all pairs S', G' with S' equivalent to S under a correspondence such that the elements of G' go into those of G. For cyclic G such systems were considered by A. A. Albert (Annals of Mathematics, (2), vol. 39 (1938), pp. 669–682). This paper generalizes his results for more general groups and considers crossed products of normal systems of semi-fields with their groups. (Received August 2, 1939.)

354. E. N. Oberg: *A note on a certain phase of best approximation.*

If $f, \phi_1, \phi_2, \cdots, \phi_n$ are linearly independent elements of a space having a norm which satisfies certain general hypotheses, and if $T(u)=f$ represents a linear functional equation whose elements are in this space, then it is shown that there exists a linear combination of the functions $\phi_1, \phi_2, \cdots, \phi_n$ which minimizes the norm of $f-T(\Phi_n)$, and that this minimizing function is in a certain sense an approximate solution of $T(u)=f$ when this equation has a unique reciprocal. One of the main purposes of this paper is to investigate the degree of approximation of the minimizing function in a subspace of the original space, an example of which is the study of uniform convergence of the polynomial determined by least mth power methods. (Received August 2, 1939.)

355. Rufus Oldenburger: *Infinite powers of matrices and characteristic roots.*

Hardy Cross introduced a method of balancing structures in civil engineering, where this method is an infinite process which (it can be shown) converges if and only if the infinite power of a real matrix exists and is zero. In the present paper necessary and sufficient conditions for the existence of the infinite power of a real or complex matrix are given in terms of characteristic roots and elementary divisors associated with this matrix. From these results it follows that a "greater than or equal to" relation can be defined for real matrices A and B so that if A and B are real matrices with nonnegative elements, and A is greater than or equal to B, the maximum absolute value of the characteristic roots of A is greater than this maximum for B. (Received July 10, 1939.)

356. L. F. Ollmann: *On cubic ring graphs.*

Cubic ring graphs were defined by E. W. Miller (see abstract 44-5-238). In this paper it is shown that any cubic graph may be imbedded in a cubic ring graph. Theorems concerning the joining of the vertices of a cubic ring graph (or a graph related to a cubic ring graph) by simple closed curves are also proved. Examples of cubic graphs are given in which, for any given number m, the vertices cannot be joined by m simple closed curves. (Received July 24, 1939.)

357. Isaac Opatowski: *Differential equations containing functionals of unknown functions.*

Application of the general implicit function theorem (T. H. Hildebrandt and L. M. Graves, Transactions of this Society, vol. 26 (1924), pp. 127–153; L. M. Graves, this Bulletin, vol. 41 (1935), p. 654; and L. Kantorovitch, Acta Mathematica, vol. 71 (1939), pp. 63–73) is madet to the solution of the following system which occurred in a problem of applied mechanics (I. Opatowski, Memorie, Accademia delle Scienze di Torino, vol. 69 (1938–1939), pp. 145–198): $d\eta_i(t, \xi)/dt = F_i[t, \xi, \xi'^1_0; \eta(t, \xi); \eta(t, \xi'^1_0)]$, where $i=1, \cdots, n$; ξ, ξ' and η mean $\xi_1, \cdots, \xi_m$; $\xi'_1, \cdots, \xi'_m$; $\eta_1, \cdots, \eta_n$, respectively; $\eta_i(t, \xi)$ are unknown functions with the initial conditions $\eta_i(0, \xi) = \bar{\eta}_i(\xi)$ and F_i are functionals eliminating ξ'. (Received July 31, 1939.)

358. G. H. Peebles: *On equivalence of certain types of series of orthonormal functions.*

This paper extends a theorem of equivalence for orthonormal polynomials in a single variable (see G. H. Peebles, Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 97–104) to somewhat more general systems of orthonormal functions, including, for example, polynomials in two real variables orthonormal on a suitable algebraic curve. (Received July 31, 1939.)

359. I. E. Perlin: *Indefinitely differentiable functions with prescribed least upper bounds.*

Let $F(x)$ be a real-valued indefinitely differentiable function of the real variable x defined on the interval $0 \leqq x \leqq a$. Let M_i denote the least upper bound of $|F^{(i)}(x)|$ on this interval. The author establishes sufficient conditions in order that there exist an indefinitely differentiable function with certain prescribed M_i. (Received July 24, 1939.)

360. B. J. Pettis: *Absolutely continuous functions in vector spaces.*

In an arbitrary space T let ξ be a Borel family of subsets that includes T and let $\alpha(E)$ be a nonnegative function completely additive over ξ, with $T = \sum T_i$ where $\alpha(T_i) < \infty$. Suppose $x(E)$, defined from ξ to a Banach space X, satisfies this condition: for each f in X^*, the adjoint of X, the numerical function $f(x(E))$ is completely additive and vanishes whenever $\alpha(E) = 0$. Then $x(E)$ is completely additive, and given $\epsilon > 0$ there is a $\delta > 0$ such that $\|x(E)\| < \epsilon$ whenever $\alpha(E) < \delta$. From this and the Vitali-Hahn-Saks theorem flow easily the chief results connected with unconditional (or complete) convergence in Banach spaces. Applications to abstract integration and linear operations also can be made. For example, let L be the space of real functions integrable over T with respect to ξ and α; then if X^* is weakly complete, every linear operation from X to L is weakly completely continuous. (Received July 31, 1939.)

361. B. J. Pettis: *The uniform differentiation of families of numerical functions.*

The paper consists of applications of results obtained previously (Duke Mathematical Journal, vol. 5 (1939), pp. 254–269). A typical theorem is the following: On $T = (0, 1)$ let $\alpha_n(t)$, $n = 1, 2, \cdots$, be a sequence of numerical functions converging everywhere and satisfying the condition $\sup_\pi \sum_{i=1}^m \sup_n |\alpha_n(t_i) - \alpha_n(t_{i-1})| < \infty$, π varying over the partitions $\pi = [t_0 = 0 < t_1 < t_2 < \cdots < t_m = 1]$ of $(0, 1)$. If the derivatives $\alpha_n'(t)$, which exist almost everywhere, also converge almost everywhere, then there is a set T_0 of measure zero such that $t \varepsilon T - T_0$ implies that $\alpha_n(t)$ is differentiable to $\alpha_n'(t)$ at t_0 uniformly with respect to n. (Received July 31, 1939.)

362. Everett Pitcher: *Critical points of a map to a circle.*

Let M be an n-dimensional manifold and F a function on M to the oriented circle. The manifold and the map are assumed to have suitable differentiability in local coordinate systems. Singular chain topology with integers modulo 2 as coefficients is used. The paper considers differential critical points and assumes that their images are isolated. A set of type numbers is assigned to each connected critical set as in the local definition in the theory of critical points of a real valued function. Let M_k denote the sum of the kth type numbers over all connected critical sets. The following theorem is proved: There are numbers Q_k, positive or zero, which are invariants of the homotopy class of F and which together with the numbers M_k satisfy the relations $M_k - M_{k-1} + \cdots + (-1)^k M_0 \geqq Q_k - Q_{k-1} + \cdots + (-1)^k Q_0$. Here k ranges from 0 to n and the equality holds when $k = n$. This result is sharper than that in the author's abstract 45-1-53. Two equivalent definitions of the numbers Q_k are given. The present method depends on the use of an *induced covering space*. (Received July 31, 1939.)

363. Everett Pitcher: *Identification of two subsets.*

Let K denote an n-dimensional complex and C_1 and C_2 separated subcomplexes which are copies of a complex C. Let L denote the complex obtained from K by identifying cells of C_1 and C_2 which are copies of the same cell of C. Let B^k denote a k-dimensional homology group (coefficients from an arbitrary group). Let N^k denote the group of *sections*, that is, homology classes of k-cycles on C for which the difference of the copies on C_1 and C_2 bounds on K. Let S^k denote the group of homology classes of k-cycles on L which are images of k-cycles on K in the map on K to L. Let T^k denote the group of homology classes on K which contain the differences of the pairs of copies of k-cycles of C. The paper proves the following theorem: I. $B^k(L) \bmod S^k(L) \approx N^{k-1}$; II. $B^k(K) \bmod T^k \approx S^k$; III. $B^k(C) \bmod N^k \approx T^k$; $k = 0, 1, \cdots, n$. Weaker results in this direction were obtained by A. B. Brown. A variety of extensions are obtained, including to the case in which C_1 and C_2 are not separated, to singular chain and Vietoris topologies, and to cocycles. (Received July 31, 1939.)

364. W. T. Puckett: *On regular transformations.*

The continuous transformation $T(M) = M'$ is said to be $(r-1)$-regular (see A. D. Wallace, abstract 44-3-16) if for every sequence of points $\{a'_n\}$ converging to a' in M' the sequence $\{T^{-1}(a'_n)\}$ converges s-regularly (see Whyburn, Fundamenta Mathematicae, vol. 25, pp. 408–426) to $T^{-1}(a')$ for every $s \leqq r-1$. If M is a continuum and a', b' are any two points of M', then the s-dimensional Betti groups of $T^{-1}(a')$ and $T^{-1}(b')$ relative to homologies in M are isomorphic for $s = 0, 1, \cdots, r$. Furthermore, in case T is a monotone 0-regular transformation, the 1-dimensional Betti group of M is the direct sum of two groups, one of which is isomorphic with the 1-dimensional Betti group of M', while the other is isomorphic with the 1-dimensional Betti group of $T^{-1}(a')$ relative to M for any point a' of M'. Thus $p^1(M) = p^1(M') + p^1(T^{-1}(a'), M)$, where $p^1(N)$ is the first Betti number of N and, for any point a' of M', $p^1(T^{-1}(a'), M)$ is the number of linearly independent 1-dimensional convergent cycles in $T^{-1}(a')$ relative to homologies in M. (Received July 10, 1939.)

365. W. T. Reid: *A theorem on continuous functions in abstract spaces.*

This note is concerned with a theorem on continuous functions in a very general abstract space, by means of which there are deduced certain results concerning semimetric spaces. In particular, the author generalizes a theorem of Montgomery (this Bulletin, vol. 40 (1934), pp. 620–624) concerning the behavior of distances between points of a metric space under transformations of the space into itself. (Received July 31, 1939.)

366. W. T. Reid: *Sufficient conditions by expansion methods for multiple integral problems of the calculus of variations.*

The present paper is concerned with sufficient conditions for a strong relative minimum for multiple integral problems of the calculus of variations, the method of proof being a refinement of the expansion method introduced for a simple integral problem by E. E. Levi. The results obtained extend those of Miranda (Memorie, Reale Accademia d'Italia, vol. 5 (1934), pp. 159–172) in not only the consideration of more general multiple integral problems, but also in the use of the Haar form of the Euler and Jacobi equations. The refinement of proof corresponds to that already given by the author in extending Levi's method to obtain sufficient conditions for the problem of Bolza, both in non-parametric and parametric form (Annals of Mathematics, (2), vol. 38 (1937), pp. 662–678; Transactions of this Society, vol. 42 (1937), pp. 183–190; also, abstract 42-9-361). (Received July 31, 1939.)

367. J. F. Ritt and H. W. Raudenbush: *Ideal theory and algebraic difference equations.*

There is derived a basis theorem for infinite systems of difference polynomials. A restricted theory of ideals is then obtained. (Received July 5, 1939.)

368. M. S. Robertson: *Typically-real functions with* $a_n=0$ *for* $n\equiv 0 \bmod 4$.

Within the unit circle the analytic function $f(z)=z+\sum_2^\infty a_n z^n$, $a_n=0$ for $n\equiv 0 \bmod 4$, takes on real values when and only when z is real. For functions of this type the author obtains the following new and sharp inequalities: For m and n both odd and $n>1$, $|a_n|+2^{-3/2}[(n-2)|a_{2m}|+n|a_2|]\leq n$, $|a_n|+2^{-1/2}(n-1)|a_2|\leq n$. For n even, $|a_n|+|a_2|\leq 2^{3/2}$, $|a_2|\leq 2^{1/2}$, $\limsup |a_n/n|\leq 1-2^{-1/2}|a_2|$. The equality signs hold in each case for the univalent function $z(1-2^{1/2}z+z^2)^{-1}=2^{1/2}\sum_{n=1}^\infty z^n \sin n\pi/4$. (Received July 31, 1939.)

369. A. E. Ross: *A note on representation of integers by positive quaternary quadratic forms.*

In this paper the representation of "large" (see Ramanujan, Proceedings of the Cambridge Philosophical Society, vol. 19 (1916), pp. 11–21; Kloosterman, Acta Mathematica, vol. 49 (1926), pp. 407–464; also cf. Dickson, *Modern Elementary Theory of Numbers*, The University of Chicago Press, 1939) integers by positive (classic) quaternary quadratic forms is studied. A canonical form with suitably chosen leading principal minors is derived and with the help of this canonical form conditions are determined under which "large" integers are represented. (Received August 1, 1939.)

370. W. E. Roth: *On the unilateral equation in matrices.* II.

Given the equation $\sum_{i=0}^{p} A_i X^i=0$, where the A_i are $m\times n$ matrices with elements in F. Solutions of this equation whose characteristic determinants are not entirely reducible to linear factors in F are obtained by operations in F. The results parallel those of an earlier paper by the author (Transactions of this Society, vol. 32 (1930), pp. 61–68). (Received August 4, 1939.)

371. A. C. Schaeffer and R. J. Duffin: *A refinement of the inequality of the brothers Markoff.*

Let $f(x)$ be a polynomial of degree n or less which satisfies the inequality $|f(x)| \leqq 1$ in the interval $-1 \leqq x \leqq 1$. The theorem of A. Markoff states that under these conditions $|f'(x)| \leqq n^2, (-1 \leqq x \leqq 1)$. The theorem of W. Markoff gives the least upper bounds for the higher derivatives of $f(x)$ This paper observes that the stated conditions for these theorems are unnecessarily restrictive. Instead of assuming that $|f(x)| \leqq 1$ at all points of the interval it is sufficient to assume that this inequality is satisfied only at the $n+1$ points $x = \cos k\pi/n$, $(k=0, 1, \cdots, n)$; and the conclusions remain unchanged. (Received August 3, 1939.)

372. Edith R. Schneckenburger: *On 1-bounding monotonic transformations which are equivalent to homeomorphisms.*

A 1-bounding transformation of a space A onto a space B is a transformation such that for each point b of B the one-dimensional Betti number of the inverse of b is zero. A transformation $T(A)=B$ is equivalent to a homeomorphism when B is homeomorphic to A. Let T represent a single-valued, continuous, 1-bounding, monotonic transformation. This paper gives: (1) a necessary and sufficient condition that T be equivalent to a homeomorphism when A is a finite graph; (2) a sufficient condition that T be equivalent to a homeomorphism when A is a tree. It is shown also that when A is a Peanian continuum imbedded in a two-manifold M and every 1-bounding monotonic T is equivalent to a homeomorphism: (a) if A is cyclic, A is a simple closed curve or the two-manifold M; (b) if A is acyclic, A is an arc; (c) if A is neither cyclic nor acyclic and certain restrictions are put upon the cut points of A, then A is a finite or countably infinite set of simple closed curves tangent to each other at a single point. (Received July 17, 1939.)

373. I. J. Schoenberg: *On local convexity in euclidean spaces.*

H. Tietze (Mathematische Zeitschrift, vol. 28 (1928), pp. 697–707) introduced the following notion of local convexity: Let M be a set in the euclidean space E_k. Let $P \varepsilon M$. The set M is said to be *locally convex at* P, if there is a positive $\rho = \rho(P)$ such that the intersection $M \cdot S(P; \rho)$ of M with the open sphere $S(P; \rho)$, of center P and radius ρ, is convex in the ordinary sense. The set M is said to be *locally convex* if it is locally convex at all its points. Tietze's chief result is the following theorem: A continuum M which is locally convex is convex in the ordinary sense. A short and simple proof of Tietze's theorem is given in this note. It operates with the notion of uniform local convexity. (Received July 31, 1939.)

374. Abraham Schwartz: *The Gauss-Codazzi-Ricci equations in Riemannian manifolds.*

The Gauss-Codazzi-Ricci equations for a $V_m \subset V_n$ are considered in the form in which they are given by J. A. Schouten and D. J. Struik (*Einführung in die neueren Methoden der Differentialgeometrie*, Groningen, 1938, vol. 2, pp. 121 ff.). It is shown that each of these equations is a consequence of others under simple conditions on the curvature affinors of valence 3. These conditions can be expressed geometrically in terms of conjugate vectors on the V_m or their analogues in the various normal spaces. Algebraically, when the normal space considered is of 1 or 2 dimensions, they can be expressed in terms of the invariants describing certain matrices, which matrices reduce to the second fundamental form in a particular case, $V_{n-1} \subset V_n$. As an illustration, the Gauss equation referring to the $(x+1)$st normal space is a consequence of the Gauss, Codazzi, and Ricci equations referring to the xth normal space and of the Ricci equation referring to the $(x+1)$st normal space if the curvature affinor referring to the xth normal space satisfies a certain condition. The result of T. Y. Thomas (*Riemann spaces of class one and their characterization*, Acta Mathematica, vol. 67 (1935), pp. 189 ff.) is included in a particular case of a theorem here. (Received July 27, 1939.)

375. A. R. Schweitzer: *On a classification of "configurational" sets of ordered dyads.*

In continuation of a previous paper reported in this Bulletin (abstract 44-11-474) configurational sets of ordered dyads are classified into types, first with reference to "connection" in the sense of the author's theory (American Journal of Mathematics, vol. 31, chap. 2) and second with reference to "right" (R) and "left" (L). The first classification distinguishes between completely connected, partially connected and completely disconnected sets. The second classification distinguishes between "R" and "L" sets in the author's genetic construction of complete systems of configurational sets by making the following conventions: 1. A set consisting of a single self-conjugate dyad is an R set. 2. An $R(L)$ set remains an $R(L)$ set after an adjunction. 3. An $R(L)$ set becomes an $L(R)$ set after a replacement. Application is made to groups of configurational sets and also to the construction of the sequence of abstract determinants: $R(\alpha\alpha)$, $R(\alpha\alpha, \beta\beta)+L(\alpha\beta, \beta\alpha)$, and so on. Reference is made to Chapter 3 of the author's theory (ibid., vol. 31), to Cayley (Journal für die reine und angewandte Mathematik, vol. 38, p. 93), and Grassmann (*Gesammelte Werke*, vols. 1–2, §504). (Received July 29, 1939.)

376. A. R. Schweitzer: *On a classification of systems of "configurational" sets of ordered dyads.*

The author classifies as follows. A. Complete systems. Represent groups. B. Incomplete systems. I. Represent groups and include systems which permit definition of finite (Galois) fields by introducing a relation of equality among the dyads of each configurational set. II_1. Do not represent groups but allow interpretation as finite geometries including pseudodescriptive geometries (American Journal of Mathematics, vol. 31, p. 379). II_2. Do not represent groups but permit definition of configurations (1) with, (2) without, reference to order and include systems which allow definition of finite nets in projective geometry. C. Systems which represent some or all of the preceding classes in relation to one another. Reference is made to the author's foundations of projective geometry (this Bulletin, abstract 42-11-443), the tactical configurations of E. H. Moore and others, and the finite geometries of Veblen and others. (Received July 29, 1939.)

377. H. A. Simmons and J. F. Paydon: *On the shortest time of fall from a curve to a point.*

A well known problem related to the present note is to determine on a curve Γ of class C', in a vertical plane, the point 1 from which a particle will fall from rest under gravity to a fixed point 2 in the shortest time. The purpose of this note is to associate with Γ, when it is mildly restricted, an envelope G such that the tangent through 2 to G intersects Γ in the point 1. The envelope G plays here a role analogous to that which is played by the envelope, when existent, of the family of normals to an arc in the problem of *finding the shortest distance from a point to an arc.* (Received July 31, 1939.)

378. Andrew Sobczyk: *Projections in Minkowski and Banach spaces.*

The results of Murray (Transactions of this Society, vol. 41 (1937), p. 138) on projections in l_p spaces, and additional quantitative information, are obtained by a briefer method. Various generalizations of l_p spaces are studied, in particular, spaces defined by a sequence $p_1, p_2, p_3, \cdots$ of exponents. For these spaces it is shown that if a projection exists on every closed linear subspace, the p_i's must converge to 2 so rapidly that the space is isomorphic to Hilbert space. A similar result is obtained for the spaces of Orlicz. (Received July 28, 1939.)

379. N. E. Steenrod: *Homology groups of the complement of a closed set in an n-sphere.*

Let S^n be an n-sphere and C a closed subset and let $S^n - C$ be subdivided into an infinite complex K. Infinite q-chains in K with coefficients in a topological abelian group G are considered. Let Z_q be the group of q-cycles, B_q the group of bounding cycles and $\overline{B}_q$ its closure (if G is compact or if C is a neighborhood retract, $B_q = \overline{B}_q$; but this is not true in general otherwise). It is well known that $\overline{H}_q = Z_q - \overline{B}_q$ is isomorphic with the Vietoris homology group of C of dimension $q-1$; it is therefore a topological invariant of C. In this paper $H_q = Z_q - B_q$ is proved to be a topological invariant of C. A method is given for defining H_q for an arbitrary compact metric C directly in terms of the topology of C. A study is made of H_1 where C is a solenoid and G is the discrete group of integers. It is proved that H_1 has the power of the continuum. This is in contrast to $\overline{H}_1 = 0$. It follows from results of Eilenberg that there is a continuum of distinct homotopy classes of maps of $S^3 - C$ into S^2. This answers a question raised by Borsuk and Eilenberg. (Received July 31, 1939.)

380. Wolfgang Sternberg: *The general limit theorem in the theory of probability.*

Let x_i, $(i = 1, 2, \cdots, n)$, be independent stochastic variables with any distribution functions V_i. Then the general limit theorem states: The distribution function $W_n(x)$ of the sum $x_1 + x_2 + \cdots + x_n$ converges (as n increases indefinitely) uniformly in x to the Gaussian function ϕ, when very general assumptions with respect to the V_i are made. Denoting standard deviation, an absolute moment of the third order of x_i, by b_i and c_i, respectively, and putting $b_1 + \cdots + b_n = B_n$ the following assumptions are made: $\lim_{n\to\infty} b_i/B_n = V$, $\lim_{n\to\infty} c_i/b_i(B_n)^{1/2} = V$ uniformly for all $i = 1, 2, \cdots, n$. A proof of the theorem is given, using the recursion formula $W_i(x) = \int_{-\infty}^{+\infty} W_{i-1}(x-\xi)dV_i(\xi)$ and the fact that the Gaussian function $\phi(x/(2y)^{1/2})$ satisfies the equation $\partial\phi/\partial y = \frac{1}{2}\partial^2\phi/\partial x^2$. One should compare this with the proof published by A. Khintchine (*Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Berlin, 1933), which is based on works of F. Petrowsky and A. Kolmogoroff. Khintchine's proof gave the suggestion for the present paper. Other assumptions have been chosen in order to make the exposition as simple as possible, while Khintchine strives for wide generality and so assumes nothing as to moments of the third order. A so-called upper- and lower-function, which Khintchine introduces as the fundamental idea of the Petrowsky-Kolmogoroffian proof, is not used by the author. (Received July 25, 1939.)

381. R. M. Thrall: *A note on groups of class* 3.

In this note, formulas are given for commutator computation in all groups of class 3. These formulas are applied to investigate the properties of groups of class 3 with all operators not equal to 1 of prime order $p>3$. If such a group has n independent generators, it is a quotient group of a particular group A_n of order p^N, $N=C_{n,1}+3C_{n,2}+2C_{n,3}$ with n independent generators. The only characteristic subgroups of A_n are the commutator subgroup and the central. The group A_n is a generalization for $p>3$ of the group A_n for $p=3$ investigated by Levi and van der Waerden (Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, vol. 9 (1933), pp. 154–158). (Received July 31, 1939.)

382. W. J. Trjitzinsky: *General theory of singular integral equations with real kernels.*

The theories of integral equations of importance from the present point of view are those due to Vólterra, Fredholm, Hilbert, E. Schmidt, Carleman. The developments of Carleman present the culminating aspects in the above sequence; moreover, his theory has already had some particularly important applications (Schrödinger wave equation, nonlinear differential equations of the type occurring in dynamics). In the present work is given an extensive and systematic treatment of equations with kernels of any finite or transfinite rank. Carleman's kernels are designated as of rank one. Kernels of rank $\alpha(>1)$ are limits (in one sense or another) of kernels of lower ranks. Thus there arises a transfinite hierarchy of theories of ascending generality. (Received July 31, 1939.)

383. C. W. Vickery: *A theorem on abstract sets.*

Suppose that S is a set, $\aleph_\alpha$ a cardinal number, and π a property such that (1) every subset of S that has property π is of power greater than or equal to $\aleph_\alpha$, (2) the collection of all subsets of S having property π is of power $\aleph_\alpha$, then for every ordinal $\beta<\alpha$ there exists a collection G_β of power $\aleph_\beta$ of mutually exclusive subsets of S, each of power $\aleph_\alpha$, no one of which contains a subset having property π and such that every subset of S that has property π contains an element of each set G_β. The proof involves Zermelo's well ordering theorem. (Received July 19, 1939.)

384. C. W. Vickery: *On spaces* (V_{ω_α}) *and spaces* Λ_{ω_α}.

A space (V) of Fréchet is said to be a space (V_{ω_α}) provided every point is of type ω_α. In order that a space (V_{ω_α}) be a space Λ_{ω_α} (C. W. Vickery, Tôhoku Mathematical Journal, vol. 40 (1934), p. 6) it is necessary and sufficient that, for each two distinct points p and q, there exist neighborhoods R_p of p and R_q of q such that $R_p\cdot R_q=0$. A generalization of condition I of Fréchet's *Les Espaces Abstraits*, p. 217, is a necessary and sufficient condition that a space Λ_{ω_α} be a space (V_{ω_α}). (Received July 19, 1939.)

385. C. W. Vickery: *On two different types of linear continua.*

Examples of several types of linear continua have been given by the author (Tôhoku Mathematical Journal, vol. 40 (1934), pp. 1–26). Space U_{1,ω_1} represents the greatest extension of the real number continuum which preserves its local properties. Consider the following axioms: (1) S is an ordered set (with respect to $\prec$) of at least two points. (2) If $a\prec b$, there exists c such that $a\prec c\prec b$. (3) Dedekind's postulate is satisfied. (4) There exists no first or last point. (5) Every point is the limit of a monotonic ascending and a monotonic descending type ω sequence of points. (6) S is compact. (6') There exist two type ω monotonic sequences of points, one ascending and the other descending, neither of which has a limit point. Axioms 1–6 are independent and characterize space U_{1,ω_1}. Axioms 1–6' are independent and characterize the real number continuum. Axioms 1–5 and the denial of 6 necessitate that S be either a ray of U_{1,ω_1} or the real number continuum. (Received July 19, 1939.)

386. C. W. Vickery: *Types of limit points in spaces* (V).

Suppose S is a space (V) of Fréchet. A point p of S is said to be of *order* $\aleph_\alpha$ provided there exists a collection V_p of power $\aleph_\alpha$ such that if M is a point set, p is a limit point of M if and only if every neighborhood of V_p contains a point of M distinct from p and provided $\aleph_\alpha$ is the smallest cardinal for which this is true. The space is said to be *uniform* at p provided there exists a monotonic descending sequence γ_p of neighborhoods of p such that (1) no point other than p is common to all elements of γ_p and (2) if R_p is a neighborhood of p, there exists an element of γ_p which is a subset of R_p; p is said to be of *type* ω_α provided ω_α is the smallest ordinal of such a sequence γ_p. If p is a point of order $\aleph_\alpha$, in order that p be of type ω_α it is necessary and sufficient that, if G_p is a collection of neighborhoods of p of power less than $\aleph_\alpha$, there exist a neighborhood of p which is a subset of every element of G_p. (Received July 19, 1939.)

387. A. D. Wallace: *On quasi-monotone transformations.*

The transformation $T(A)=B$ is said to be quasi-monotone provided that for each open connected set R of B the set $T^{-1}(R)$ has only a finite number of components each of which maps onto all of R. If A is a locally connected continuum then T is quasi-monotone if and only if it is the product of a monotone and a light interior transformation. It follows that a topological property is invariant under quasi-monotone transformations if and only if it is invariant under both monotone and light interior transformations. Certain results of a previous communication (abstract 45-1-69) are extended from monotone to quasi-monotone (and hence to interior) transformations. (Received July 28, 1939.)

388. M. S. Webster: *Note on certain Lagrange interpolation polynomials.*

The fundamental Lagrange interpolation polynomial $l_k^{(n)}(x) \equiv \phi_n(x)/[(x-x_k)\phi_n'(x_k)]$, where $\phi_n(x) \equiv (x-x_1)(x-x_2) \cdots (x-x_n)$ and $-1<x_k<1$, is considered when $x_1, x_2, \cdots, x_n$ are the zeros of the Jacobi polynomial $\phi_n(x;\,\alpha,\,\beta)$ of nth degree which satisfies the differential equation $(1-x^2)\phi_n''(x)+[\alpha-\beta-(\alpha+\beta)x]\phi_n'(x)+n(n+\alpha+\beta-1)\phi_n(x)=0$. For special values of the parameters α, β, the least upper bound for $|l_k^{(n)}(x)|$ in $(-1, 1)$ is obtained. For example, if $\alpha=\beta=3/2$, the least upper bound is 2. Certain limiting relations are also given. (Received June 24, 1939.)

389. G. W. Whaples: *On the structure of moduls with a commutative algebra as operator domain.* Preliminary report.

The solutions T of the matrix equations $TA_i=B_iT$, $(i=1, \cdots, n)$, over field F, correspond to certain homomorphisms of the total vector space with left operators $F[I, A_1, \cdots, A_n]$ to subspaces of the total vector space with left operators $F[I, B_1, \cdots, B_n]$. A new method for determining the solutions of $TA_i=B_iT$, some theorems on the possible ranks of these solutions, and a representation for the ring of all matrices commutative with $A_1, \cdots, A_n$ are given for the case in which $F[I, A_1, \cdots, A_n]$ and $F[I, B_1, \cdots, B_n]$ are commutative. If R is a commutative algebra with unit over an infinite separable field F, the minimum n such that $R=F[r_1, \cdots, r_n]$ is found; R is a principal ideal ring if and only if $n=1$. Many of the theorems on R-modules which are true for $n=1$ cannot be extended to the case $n>1$; for example, if $n>1$, R-modules are not always expressible as direct sums of cyclic R-modules. (Received August 3, 1939.)

390. P. M. Whitman: *Free lattices.*

It is proved that in the free lattice generated by a set of elements X_i, (1) $X_i \leqq X_j$ if and only if $i=j$, and (2) recursively, $A \leqq B$ if and only if one or more of the following are true: (a) $A \equiv A_1 \cup A_2$, where $A_1 \leqq B$ and $A_2 \leqq B$, (b) $A \equiv A_1 \cap A_2$, where $A_1 \leqq B$ or $A_2 \leqq B$, (c), (d) the dual conditions. A canonical form for the elements of such a lattice is deduced, and it is shown that if the number of generators is finite then certain elements cover others. (Received June 26, 1939.)

391. G. T. Whyburn: *A relation between interior and non-alternating transformations.*

Theorems are first established from which it follows that an interior transformation of a compact continuum into a dendrite will be non-alternating if and only if the inverse of each end point of the dendrite is connected. It is then shown that if M is a compact continuum and H is the set of all end points of a dendrite D, then any interior transformation $f(M)=D$ such that the collection $[f^{-1}(p)]$, $p \in H$, is semi-closed can be factored into the form $f=f_2f_1$ where f_1 is equivalent to f on $f^{-1}(H)$ and topological on $M-f^{-1}(H)$ and f_2 is interior and non-alternating. (Received June 30, 1939.)

392. G. T. Whyburn: *On the interiority of real functions.*

In this paper it is shown that if $f(x)$ is a real-valued continuous function defined on a connected and locally connected separable metric space M (therefore transforming M into a set Y of real numbers), there exists a countable subset C of Y such that f is interior at every point of $M-f^{-1}(C)$. Also the following extension theorem is proven: If M is a subcontinuum of a cyclic locally connected compact continuum L and $f(M)=(0, 1)$ is continuous and such that for any y, $0 \leqq y \leqq 1$, $L-f^{-1}(y)$ is connected, there exists an extension $\phi(L)=(0, 1)$ of f to L which is interior at every point of $L-M$ and at every point of M where f is interior. (Thus if f is interior on M, ϕ is interior on L.) (Received June 30, 1939.)

393. L. R. Wilcox: *A new foundation for projective differential geometry.*

It was recently proved by von Neumann that the n-dimensional projective space L_n over a field F is isomorphic to the lattice of all (principal) right ideals in the ring F_n of all nth order matrices with elements in F. This paper assumes F to be real or complex and lays the foundations for a projective differential geometry of L_n wholly in terms of the algebraic and topological structure of F_n, the topology of F_n being that obtained by defining for $A \in F_n$ a norm $\|F\| \equiv (\text{trace } (A^*A))^{1/2}$. It is shown that this theory leads to classical projective differential geometry; differential equations of a configuration are found to appear as conditions on the trace of certain functions of the matrix representing the configuration. (Received August 1, 1939.)

394. Margarete C. Wolf: *Transformations of bases for relative sets over a non-commutative field.*

This is a continuation of a paper by the author (this Bulletin, vol. 44 (1938), pp. 716–718) for the case that R is a vector space of dimension n over a non-commutative field $\mathfrak{F}$, and M a matrix of a linear operator relative to a base of R. If $\xi_1, \xi_2, \cdots, \xi_k$ is a set of vectors such that R is a direct sum of the cyclic subspaces $L_M(\xi_i)$, $(i=1, 2, \cdots, k)$, the problem of finding conditions under which R can be expressed as a direct sum of other cyclic subspaces has been made to depend upon the unique solution of k sets of n linear equations in n unknowns with coefficients in $\mathfrak{F}$. In special cases several more specific results are obtained. For example, if $L_M(\xi_1) \cap L_M(\xi_2) = 0$ and if ξ_1 and ξ_2 are minimally associated relative to M with polynomials g_1 and g_2 respectively, then a necessary and sufficient condition that there exist a vector η such that $L_M(\eta) = L_M(\xi_1) \oplus L_M(\xi_2)$ is that g_1 and g_2 be similar to a pair of relatively prime polynomials. (Received July 25, 1939.)

395. J. W. T. Youngs: *Arc-spaces.*

Certain axioms in terms of *point* and *arc* determine arc-spaces. Cyclic elements are defined and it is noticed that the hyperspace of cyclic elements is itself an arc-space which is a dendrite. The development is analogous to that of cyclic element theory for Peano spaces, and one sense in which the hyperspace of cyclic elements of a Peano space may be said to be a dendrite is that it is an arcspace with one and only one arc joining any two of its points. (Received July 24, 1939.)

396. J. W. T. Youngs: *Generalized cyclic elements.*

If $C(ab)$ is the cyclic chain from a to b, then $C(ab)$ is a true cyclic element if and only if a is conjugate to b. The generalization to true k-cyclic elements is made from this point of view. A point a is said to be k-conjugate to b if no set of k distinct points separates a from b in the space. Roughly speaking, if $a_1, \cdots, a_{k+1}$ are distinct, and k-conjugate to each other then the totality of points k-conjugate to a_i for $=1, \cdots, k+1$, constitute a true k-cyclic element. In spite of the fact that a k-cyclic element is not, in general, connected, a large number of the desirable properties of cyclic elements may be extended to k-cyclic elements. (Received July 24, 1939.)