# A TEST-RATIO TEST FOR CONTINUED FRACTIONS*

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**Introduction.** The general question of convergence of continued fractions of the form $1+K_1^\infty[b_n/1]$ remains in a large measure unanswered, even though continued fractions of this type are of especial importance from a function-theoretic point of view. Valuable contributions have been made by E. B. Van Vleck, A. Pringsheim, O. Szász, O. Perron, and others. Leighton and Wall [7] recently gave new types of convergence criteria for continued fractions of this kind. Jordan and Leighton in a paper to be published soon give a large number of new sets of sufficient conditions for convergence.

The purpose of the present paper is to establish the first test-ratio test for continued fractions and a very general theorem on convergence, which is also believed to be the first of its kind. This test leads to a class of continued fractions, the *precise* region of convergence of which is the interior of a circle. This is a new phenomenon.

1. **A test-ratio test.** Let

$$(1.1) \qquad 1 + \mathop{K}_1^\infty [b_n/1] = 1 + \frac{b_1}{1} + \frac{b_2}{1} + \cdots$$

be a continued fraction in which the $b_n$ are complex numbers $\neq 0$.

THEOREM 1. *If the ratio $|b_{n+1}/b_n|$ is less than or equal to $k<1$ for $n$ sufficiently large, the continued fraction* (1.1) *converges at least in the wider sense. If $|b_{n+1}/b_n|$ is greater than or equal to $1/k>1$ for $n$ sufficiently large, the continued fraction diverges by oscillation. If the limit of the ratio is unity, the continued fraction may converge or diverge.*

Suppose $|b_{n+1}/b_n| \leqq k<1$ for $n$ sufficiently large. It follows that there exists a positive integer $N$ such that $|b_n| <1/4$ for $n \geqq N$. Each continued fraction $K_n^\infty[b_n/1]$ then converges (Van Vleck [2], Pringsheim [4]) for $n \geqq N$. The proof of the first statement of the theorem is complete.

Assume $|b_{n+1}/b_n| \geqq 1/k>1$ for $n$ sufficiently large. Write (1.1) in the equivalent form (Perron [8], p. 197)

$$(1.2) \qquad 1 + \mathop{K}_1^\infty [1/a_n],$$

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where

$$a_{2n} = \frac{b_1 b_3 \cdots b_{2n-1}}{b_2 b_4 \cdots b_{2n}}, \quad a_{2n-1} = \frac{b_2 b_4 \cdots b_{2n-2}}{b_1 b_3 \cdots b_{2n-1}}, \quad n = 1, 2, 3, \cdots; b_0 \equiv 1.$$

It will be shown that the series $\sum |a_n|$ converges, and it will follow from a theorem of Stern [1] that the continued fraction (1.2), and hence (1.1), diverges by oscillation. It is sufficient to observe that

$$|a_{2n}/a_{2n-2}| = |b_{2n-1}/b_{2n}| \leqq k < 1,$$
$$|a_{2n+1}/a_{2n-1}| = |b_{2n}/b_{2n+1}| \leqq k < 1,$$

for $n$ sufficiently large. Thus the two series $\sum |a_{2n+1}|$ and $\sum |a_{2n}|$ converge. It follows that the series $\sum |a_n|$ converges, and the second statement of the theorem follows as indicated.

To prove the final statement of the theorem, it is sufficient to consider the example

$$(1.3) \qquad 1 + \frac{a}{1} + \frac{a}{1} + \cdots.$$

When $a = 1$ it is well known that this continued fraction converges to the value $(1+5^{1/2})/2$. When $a = -1$, a computation of the successive approximants proves immediately that the continued fraction diverges. Indeed, Szász [6] has shown that the continued fraction (1.3) diverges for every $\epsilon > 0$, if $a = -\epsilon - 1/4$.

Corollary. *If* $\lim_{n\to\infty} |b_{n+1}/b_n| = k$, *the continued fraction* (1.1) *will converge, at least in the wider sense, if* $k < 1$, *and will diverge if* $k > 1$.

The proof is immediate.

Example. *A continued fraction with a circle as its region of convergence.* Consider the continued fraction

$$(1.4) \qquad 1 + \mathop{K}_{1}^{\infty}[c_n x^n/1] = 1 + \frac{c_1 x}{1} + \frac{c_2 x^2}{1} + \cdots,$$

where the $c_n$ are complex nonzero numbers. If $\lim_{n\to\infty} |c_{n+1}/c_n| = c \neq 0$, it follows from the preceding corollary that the continued fraction (1.4) converges, at least in the wider sense, to a function analytic except possibly for a finite number of poles in every closed region wholly interior to the circle $|x| = 1/c$, and diverges outside. Further, if $\lim_{n\to\infty} |c_{n+1}/c_n| = 0$, (1.4) converges to a function meromorphic throughout the finite plane.

2. **A general theorem on convergence.** Leighton and Wall [7] gave an example of a convergent continued fraction (1.1) where the elements $b_n$ were everywhere dense in the complex plane. The following theorem attacks the general question of convergence from a different point of view. We assume as usual that all $b_n \neq 0$.

THEOREM 2. *Let $m_0, m_1, m_2, \cdots$ be any sequence of positive integers such that $m_0 = 2$, $m_{n+1} - m_n \geqq 2$, $(n = 0, 1, 2, \cdots)$. The numbers*

$$b_{m_0}, b_{m_1}, b_{m_2}, \cdots \tag{2.1}$$

*can be chosen in such a fashion that with at most one value in the complex plane excluded from each of the numbers $b_n$ not contained in the set* (2.1), *the continued fraction* (1.1) *will converge.*

Let $A_n/B_n$ represent the $n$th approximant of (1.1), where $A_n$ and $B_n$ are given by the usual recursion relations

$$\begin{aligned} &A_0 = 1, \qquad B_0 = 1, \qquad A_1 = 1 + b_1, \qquad B_1 = 1, \\ &A_n = A_{n-1} + b_n A_{n-2}, \\ &B_n = B_{n-1} + b_n B_{n-2}, \qquad\qquad n = 2, 3, 4, \cdots . \end{aligned} \tag{2.2}$$

By means of (2.2) write $A_j$ and $B_j$, $(j = 2, 3, \cdots, m_1 - 1)$, as

$$A_j = f_0^j A_1 + b_2 g_0^j A_0, \qquad B_j = f_0^j B_1 + b_2 g_0^j B_0, \tag{2.2'}$$

where $f_0^j$ and $g_0^j$ are polynomials in the numbers $b_3, b_4, \cdots, b_{m_1-1}$ and do not depend on any other $b$'s. (Perron [8], p. 14, uses the symbol $A_{t-m_r,m_r}$ for $f_r^t$, and $B_{t-m_r,m_r}$ for $g_r^t$). Suppose the numbers $f_0^j$ are nonzero. It is clear that $|b_{m_0}| = |b_2|$ can be chosen so small that simultaneously

$$\left| \frac{A_j}{B_j} - \frac{A_1}{B_1} \right| < \frac{1}{2}, \qquad j = 2, 3, \cdots, m_1 - 1.$$

Now write $A_k$ and $B_k$, $(k = m_1, m_1 + 1, \cdots, m_2 - 1)$, as

$$\begin{aligned} A_k &= f_1^k A_{m_1-1} + b_{m_1} g_1^k A_{m_1-2}, \\ B_k &= f_1^k B_{m_1-1} + b_{m_1} g_1^k B_{m_1-2}, \end{aligned} \tag{2.2''}$$

where $f_1^k$ and $g_1^k$ are polynomials in $b_{m_1+1}, b_{m_1+2}, \cdots, b_{m_2-1}$ and do not depend on any $b$'s not in this set. Similarly, let us suppose for the moment that the numbers $f_1^k$ are never zero. The number $|b_{m_1}|$ can then be taken so small that

$$\left| \frac{A_k}{B_k} - \frac{A_{m_1-1}}{B_{m_1-1}} \right| < \frac{1}{2^2}, \qquad k = m_1, m_1 + 1, \cdots, m_2 - 1.$$

Continue the process. With the assumption that $f_r^t$ is never zero it is clear that $|b_{m_r}|$ can be chosen so small that

$$\left| \frac{A_t}{B_t} - \frac{A_{m_r-1}}{B_{m_r-1}} \right| < \frac{1}{2^{r+1}}, \qquad t = m_r, m_r + 1, \cdots, m_{r+1} - 1.$$

The continued fraction will thus converge.

It remains to assign conditions to the numbers $b_n$ so that the numbers $f_r^t$ will be different from zero. It is sufficient to exclude precisely one value in the finite complex plane from each $b_n$ not in the set $b_{m_0}, b_{m_1}, \cdots$. For, in the general case, it follows from (2.2) that

$$\begin{aligned} &f_r^{m_r} = 1, \qquad f_r^{m_r+1} = 1 + b_{m_r+2}, \\ &f_r^{m_r+s} = f_r^{m_r+s-1} + b_{m_r+s} f_r^{m_r+s-2}, \quad s = 2, 3, \cdots, m_{r+1} - m_r - 1, \end{aligned} \tag{2.3}$$

where $f_r^{m_r+s-1}$ is a polynomial in $b_{m_r+2}, b_{m_r+3}, \cdots, b_{m_r+s-1}$ and depends on no other $b$'s. The value $-1$ is first excluded from $b_{m_r+2}$. It follows from (2.3) that one value may be excluded from each successive $b$ in such a way that $f_r^t$ is never zero. This completes the proof of the theorem.

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