ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

167. R. P. Agnew: *On oscillations of real sequences and their transforms by square matrices.*

The oscillation $\Omega(s)$ of a sequence s_n of complex numbers is defined by $\Omega(s) = \limsup_{m,n\to\infty} |s_m - s_n|$, and the transform y_n of a sequence x_n by a matrix $A \equiv \|a_{nk}\|$ of complex constants is defined by $y_n = \sum_{k=1}^{\infty} a_{nk}x_k$, $(n=1, 2, \cdots)$. This paper gives several characterizations of the matrices A having the property that $\Omega(y) \leqq \Omega(x)$ for each bounded real divergent sequence x_n. (Received March 5, 1938.)

168. A. A. Albert: *A note on normal division algebras of prime degree.*

The note gives a proof of the following theorem: Let D be a normal division algebra of degree p over K and let m be prime to p. Then if D has a normal splitting field W of degree mp over K with a cyclic subfield of degree m, the algebra D is cyclic. The result is seen to be a generalization of and to provide a new proof for the theorem that all normal division algebras of degree three over K of characteristic three are cyclic. (Received March 4, 1938.)

169. C. B. Allendoerfer: *An indicatrix for Riemann spaces.*

The spherical indicatrix of an n-dimensional Riemann space imbedded in an m-dimensional euclidean space is defined as a generalization of Gauss' indicatrix. This permits a discussion of the lines of curvature and the total curvature of such a space. It is proved that when the dimension of the indicatrix is r, there exist $n-r$ independent families of straight lines on the surface. Applications to special cases are given. (Received March 11, 1938.)

170. H. A. Arnold: *Fixed-point theorems in semi-ordered spaces.*

A comparative study is made of the fixed-point theorems of Tychonoff, Leray and Schauder, and Rothe, and generalizations are deduced for semi-ordered linear spaces. The calculus of Fréchet differentials in semi-ordered linear spaces, developed in a previous paper, is applied to obtain a generalization of a theorem of Leray and Schauder. (Received March 12, 1938.)

171. H. A. Arnold: *The theory of integration in linear L-spaces.*

The author studies the theory of functions on R to L^*, where R is the space of real numbers and L^* is a general linear space endowed with a notion of limit. A Riemann integral is defined, by the use of the idea of a transfinite sequence of sets, and is proved to exist and to be unique when the integrand is a continuous function. The fundamental theorem of the integral calculus is extended to this integral, where the derivative is a generalization of that of Gateaux. The Fréchet differential of a functional relation from one L^* space to another is defined, and it is proved that a necessary and sufficient condition that such a relation be constant over a convex set is that its Fréchet differential be zero. (Received March 12, 1938.)

172. W. L. Ayres: *On point-wise almost periodic homeomorphisms.*

A transformation T is said to be periodic in the large on the set A if $T^n(A)=A$ for some n. The transformation is said to be element-wise almost periodic if for any $\epsilon>0$ there exists an n such that for each cyclic element E there is a cyclic element E' so that $T^n(E')=E'$ and $E+T^n(E)\subset S(E',\epsilon)$. In this note it is shown that a point-wise almost periodic homeomorphism T of a Peano space M is element-wise almost periodic and is periodic in the large on every cyclic element of M, save possibly its end points. (Received March 18, 1938.)

173. G. A. Baker: *Correlation surfaces of two or more indices when the components of the indices are normally distributed.*

The distribution of a single index, both of whose components follow the normal law, was given in a paper, *Distribution of the means divided by the standard deviations of samples from non-homogeneous populations*, Annals of Mathematical Statistics, February, 1932, pp. 3–5. The present paper gives the simultaneous distributions of two or more indices when all components follow the normal law. The original frequency function of the components is transformed in terms of indices and the denominator component. The denominator component is then eliminated by integration. In addition it is shown that both components and indices may be uncorrelated over restricted ranges even though the indices are usually correlated when the components are not. (Received February 16, 1938.)

174. R. H. Bardell: *The inequalities of Morse when the maximum type is at most three.*

In this paper the inequalities of Morse, for the extremal arcs joining two fixed points in the plane, are obtained. It is assumed that each extremal arc is of type $n\leqq 3$. The method of proof is to arrange the extremal arcs joining the two fixed points into pairs such that the extremal arcs of each pair differ in type by unity. Let E_0 designate an extremal arc of type zero joining the two fixed points. Then the extremal arcs of a pair are defined to be those for which a given intersection with E_0 moves on and off E_0. This method is of interest since all previous treatments of the problem either have made use of rather complicated theorems from analysis situs or have assumed reversibility of the extremal arcs. (Received March 10, 1938.)

175. R. C. F. Bartels: *Saint-Venant's flexure problem for a regular polygon.* Preliminary report.

The paper gives a solution of the flexure problem for a cylindrical beam with cross section in the form of a regular polygon of any number of sides. The problem requires the solution of a Dirichlet problem for the polygonal region. The boundary conditions are determined from the equation $X_z \cos(x, n)-Y_z \cos(y, n)=0$ that the components of shearing-stress X_z and Y_z satisfy on the lateral surface of the beam, n representing the externally directed normal to the boundary of the cross section. The method used was devised by N. Muschelisvilli (Rendiconti della Reale Accademia dei Lincei, (6), vol. 9 (1929), pp. 295–300). It consists of mapping the region in question upon a unit circle and employing an integral formula analogous to that of Schwarz. (Received March 11, 1938.)

176. E. T. Bell: *Generalized Stirling transforms of sequences.*

From any initial sequence $A_n^{(0)}$, $(n=0, 1, \cdots)$, an infinity of sequences $A_n^{(r)}$, $(r=0, \pm 1, \pm 2, \cdots)$, are generated by the process of transformation $A_n^{(r)} \equiv (A^{(r+1)})_n$, where $(x)_n$ denotes the factorial function $x(x-1)\cdots(x-n+1)$, $(n>0)$, $(A^{(r)})_0 \equiv A_0^{(r)}$, and, after expansion of the symbolic factorial, the sth power of $A^{(r+1)}$ is replaced by $A_s^{(r+1)}$. Hence, if the $A_n^{(0)}$ are integers, Lagrange's identical congruence and its extensions at once give congruence properties of the $A_n^{(r)}$. A triple infinity of integer constants, generalizing the Stirling numbers of both kinds, appear as coefficients, whatever the initial sequence; these are also investigated. (Received March 14, 1938.)

177. B. A. Bernstein: *Sets of postulates for Boolean groups.*

The elements of a Boolean algebra form a group with respect to the operation o given by $ab'+a'b$. The author gives a number of sets of postulates for this *Boolean* group. These postulate sets bring out the relation of o to the "$+$" and the "$-$" of abelian groups. (Received March 12, 1938.)

178. Garrett Birkhoff: *New properties of lattices.*

It is shown that the sublattice of any modular lattice generated by two chains is distributive; this is related to the Schreier-Zassenhaus generalization of the theorem of Jordan-Hölder. Again, the decomposition of any partially ordered system with 0 and I into direct factors, is strictly unique; this was previously known only for lattices. (Received March 15, 1938.)

179. R. P. Boas (National Research Fellow): *On the Stieltjes moment problem.* Preliminary report.

The author considers the generalized Stieltjes moment problem $\mu_n = \int_0^\infty t^{\lambda_n} d\alpha(t)$, where $\alpha(t)$ is normalized and non-decreasing, $0 \leqq \lambda_0 < \lambda_1 < \cdots$, and $\lambda_n \to \infty$. Conditions on the sequence $\{\mu_n\}$ sufficient for the function $\alpha(t)$ (if it exists) to be unique are obtained by studying the function $f(z) = \int_0^\infty t^z d\alpha(t)$ in the half-plane $\Re(z) > 0$. When applied to the special exponents $\lambda_n = n$, the present methods give only $\lim_{n\to\infty} n^{-1}\mu_n^{1/(2n)} = 0$ as sufficient for the moment problem to be determined, and not the better criterion of Carleman, that is, the divergence of $\sum_{n=0}^{\infty} \mu_n^{-1/(2n)}$. The known methods for establishing Carleman's criterion do not appear to be applicable to general sequences $\{\lambda_n\}$. (Received March 17, 1938.)

180. Richard Brauer: *On groups of linear transformations.*

The decomposition of the regular representation of an algebra into irreducible, maximal completely reducible, and indecomposable constituents is studied, and a number of new theorems are obtained. The method used is elementary and based on an extension of Schur's lemma. It yields at the same time proofs of the well known theorems of Burnside, Frobenius, Schur, and Wedderburn. (Received March 18, 1938.)

181. J. L. Brenner: *The group of automorphisms of the abelian group of order p^{nr} and type $(r, r, \cdots, r)$.*

In a paper to appear in the Annals of Mathematics all the normal subgroups are found in the multiplicative group of n^2 matrices (a_{ij}), a_{ij} a residue class (mod p^r), $|a_{ij}| \not\equiv 0$ (p). This group $\mathfrak{G}_{p,n,r}$ is isomorphic with the group mentioned in the title of this abstract. The article of which this is an abstract describes the normal subgroups $\mathfrak{N}$ of $\mathfrak{G}$ from a group-theoretic standpoint. Let $\mathfrak{N}_s$ be the smallest group in $\mathfrak{G}$ which fixes every element of order $\leqq p^s$ and no others in the fundamental abelian group $\mathfrak{A}$; let $\mathfrak{N}_s{}^*$ be the largest group in $\mathfrak{G}$ which fixes all cyclic groups of order $\leqq p^s$ in $\mathfrak{A}$. Then if $p>2$, we have for suitable $s=s(\mathfrak{N})$: $\mathfrak{N}_s^* \supseteq \mathfrak{N} \supseteq \mathfrak{N}_s$ where $\mathfrak{N}_s^*/\mathfrak{N}_s$ *is the direct product of cyclic groups of orders* a, b, *where* $b|a$, $b=(u, v)$, $a=(p^r-p^{r-1})/b$, $u=(n, p^s-p^{s-1})$, $v=p^{r-s}$. If $p=2$, $n>2$, $r \geqq 2$, then $\mathfrak{N}_s^*/\mathfrak{N}_s$ *is the direct product of cyclic groups of orders* a, b, 2, *where* $a=2^{r-2}/b$, $b=(n, 2^{r-s}, c)$, $c=\max(2^{s-2}, 1)$. There is a corresponding theorem for $n=p=2$ involving at times the direct product of five cyclic groups and at other times a non-abelian group. (Received March 17, 1938.)

182. E. T. Browne: *On the classification of correlations in space.*

This paper is concerned with a projective classification of non-singular correlations in space. By choosing the matrix in a special canonical form, similar to that employed in a recent paper (American Mathematical Monthly, vol. 44 (1937), pp. 566–573) by the author and C. A. Denson in classifying correlations in the plane, it is shown that a classification of space correlations can be made quite simply without invoking the theory of elementary divisors. (Received March 15, 1938.)

183. H. H. Campaigne: *The properties of finite hypergroups.*

The notion of hypergroup is a generalization of that of group. If there is an equivalence relation among the elements of a hypergroup, the resulting classes are proved to constitute a hypergroup. Evidently there is at least one hypergroup H for which there exists a group G such that H is simply isomorphic to the hypergroup of classes of G. The question was raised by Marty in 1934 as to whether there exists such a group for every hypergroup, or even for every normal hypergroup. In this paper it is proved that such is not the case. However, it is shown that every hypergroup is a hypergroup of classes of elements of another hypergroup. Some additional results are obtained, among them an analog of Lagrange's theorem. (Received March 10, 1938.)

184. A. F. Carpenter: *Involutory systems of curves on ruled surfaces.*

In a paper presented to the International Mathematical Congress in Toronto in 1924, N. B. MacLean discussed the properties of a certain one-parameter family of curves lying on a ruled surface and characterized by the condition that they form a constant cross ratio with the complex curves of the surface. It is the purpose of the present paper to generalize MacLean's system of curves and to call attention to certain interesting special cases. (Received March 7, 1938.)

185. J. W. Cell: *An accurate method for obtaining the derivative function from observational data.*

Three to five independent determinations of the value of the derivative for a particular value of the independent variable are made by the use of a "sliding" quartic, a method due to Rutledge. A weighted average of these determinations is then obtained. The weights are shown to be the coefficients in the partial derivative of order $p+q$, (p with respect to α, q with respect to β, $0 \leqq p+q<5$), of the function $\sum_{i=0}^{5} C_{i,5}^{2}\alpha^{5-i}\beta^{i}$. Estimates for the error are also found. The form of the weights suggests a theorem about the weighted average of $n+1$ determinations obtained by the use of a "sliding" n-ic, and this theorem is established. (Received March 6, 1938.)

186. W. S. Claytor and R. L. Wilder: *Homogeneous locally connected continua.* I.

It was shown by Mazurkiewicz (Fundamenta Mathematicae, vol. 5 (1924), pp. 137–146) that a compact, locally 0-connected, homogeneous continuum in the plane is a simple closed curve. The purpose of the present paper is to consider the higher dimensional analogs of this theorem. It is shown that if M is a two-dimensional, compact, locally 0- and 1-connected (in terms of Vietoris cycles) continuum in euclidean three-space, then M is a closed two-dimensional manifold. The second paper will treat the general n-dimensional case. (Received March 11, 1938.)

187. P. H. Daus: *Correlations in terms of central collineations and central correlations.*

In a previous communication, the author considered the geometric constructions and the classification of collineations as determined by a chain of central collineations. In this paper a central correlation, taken as the simplest correlation, is defined as one determined by a center S, axis s, such that if A and a are correlated elements, then a, s, SA are concurrent. It is shown how the correlation determined by four sets of correlated elements may be constructed linearly by a chain of three simple transformations, which are central correlations or central collineations in different orders. The lemmas used in the constructions also furnish a geometric basis for the classification of correlations. (Received March 11, 1938.)

188. R. P. Dilworth: *Non-commutative residuation.* Preliminary report.

In a lattice Σ, completely closed with respect to union, over which a non-commutative multiplication is defined, there exists a right (left) residual $a \cdot b^{-1}$ ($b^{-1} \cdot a$) having the properties: $a \supset (a \cdot b^{-1})b$, $a \supset xb$ implies $a \cdot b^{-1} \supset x$ ($a \supset b(b^{-1} \cdot a)$, $a \supset bx$ implies $b^{-1} \cdot a \supset x$). The commutative law for residuation does not hold in general. The study of such domains has been begun by Krull, and the structure in the commutative case has been treated in detail by Morgan Ward and the author. In the present paper the relationship between the multiplication and residuation is considered in detail. Structure theorems are proved analogous to those in the commutative case, and a start is made in extending Krull's investigations on the arithmetic properties of these domains. (Received March 11, 1938.)

189. C. H. Dix: *Matrix expressions of Hooke's law.*

The generalization of Hooke's law, namely, a set of six linear equations with thirty-six coefficients connecting the six stress components with the six strain components, given in works on elasticity, is unsymmetrical and, at least in the case of the isotropic medium, unnecessarily complicated. Instead of the six vectors, consider the corresponding nine term square matrices. These matrices are defined so that a rotation R carries a stress matrix into the new stress matrix referred to the rotated coordinates and has a similar effect on the strain matrix. On principal axes the law is $S_i=2\mu s_i+\lambda\theta$, $(i=1, 2, 3)$, so that on principal axes $S_p=2\mu s_p+\lambda\theta I$, where S_p, s_p, I are stress, strain, and unit matrices, respectively, and θ is the spur of s_p. Evidently, then, to obtain the general relation (isotropic case) one merely performs the rotation R, obtaining $RS_pR^{-1}\equiv S\equiv 2\mu Rs_pR^{-1}+\lambda\theta RIR^{-1}\equiv 2\mu s+\lambda\theta I$. (Received March 11, 1938.)

190. C. H. Dix: *Symmetrical form for the general strain matrix.*

The general strain matrix including second order terms can be put in the symmetric form $s=(V+V^*+VV^*)/2$, where the terms of V are $u_{x_j}^i$'s, $x_i'=x_i+u^i(x_1, x_2, x_3)$, and V^* is the transpose of V. This form is especially convenient for use in writing the dynamical equations in matrix form. (Received March 11, 1938.)

191. D. M. Dribin (National Research Fellow): *Prüfer ideals in commutative rings.*

Let $\mathfrak{R}$ be a commutative ring with a unit and having no divisors of zero; let $\mathfrak{g}$ (a set of integral elements) be a subring of $\mathfrak{R}$ and $\mathfrak{o}$ a subring of $\mathfrak{g}$. To every finite set of elements of $\mathfrak{g}$, $a_1, \cdots, a_n$, will correspond an *ideal* $(a_1, \cdots, a_n)$ in $\mathfrak{R}$, defined as any set of elements of $\mathfrak{g}$ that satisfies certain postulates. Different ideal systems define ideals by different correspondences, but in any ideal system $\mathfrak{M}$ the (principal) ideal (a) consists of all multiples of a by elements of $\mathfrak{o}$. In $\mathfrak{M}$ there may be defined a series of five properties A, $\cdots$, E, that an ideal may possess; for example, $\mathfrak{M}$ has property A if every ideal defined by $\mathfrak{M}$ is principal. In the present paper three ideal systems $\mathfrak{L}$, $\mathfrak{A}$, $\mathfrak{B}$, previously studied by Prüfer for the case of fields (Journal für die reine und angewandte Mathematik, vol. 168 (1932), pp. 1–36) but with a less general definition of ideal, are considered, and the properties A, . . . , E for each are studied, as well as the equivalences of properties in different ideal systems. (Received March 14, 1938.)

192. R. J. Duffin: *Structure elements of quasi-groups.*

A quasi-group may be defined as a set of symbols having a multiplication table in which each symbol occurs once and only once in every row and in every column. In a recent paper Hausmann and Ore (American Journal of Mathematics, vol. 59 (1937), p. 983) have studied quasi-groups with various associative laws. In this paper no associative law whatever is assumed. The theorems demonstrated are mainly concerned with those subsets which are associative as a unit with every pair of elements of the quasi-group. This is analogous to the introduction of commutative sets in the study of non-commutative groups. The order of a subquasi-group which is right-associative in the above sense divides the order of the group. Those subquasi-groups which are both commutative and associative as a unit are shown to form a Dedekind structure. The theorem of Jordan-Hölder may be stated in exactly the same way as for groups and includes the theorem for groups. (Received March 10, 1938.)

193. Ben Dushnik: *Concerning continuous linear orders.*

Let M be a continuous linearly ordered set. (1) Let D be a nowhere dense subset of M which is the complement of a set of non-abutting intervals in M. Then a necessary and sufficient condition for D to be similar (as a type of order) to the Cantor discontinuum is that every set of non-overlapping intervals in M is at most denumerable. (2) Let f be a function defined on all of M with "values" in M. An element $c\subset M$ such that $f(c)=c$ is called a fixed point of M (with respect to f). Necessary and sufficient conditions are given for the existence of fixed points in case (a) f is continuous, or

(b) f is an almost-similarity function, that is, in case $x<x_1$ implies $f(x) \leqq f(x_1)$. (Received March 11, 1938.)

194. J. M. Earl: *The convergence of polynomials on an infinite interval.*

Let $f(x)$ be a given function of x defined on an infinite interval. Let polynomials $P_n(x)$, $(n=0, 1, 2, \cdots)$, be determined so that the integral of the product of a given non-negative weight function $r(x)$ and the mth power of the error is a minimum. When $f(x)$ and $r(x)$ are restricted suitably, an upper bound for the product $r(x)^{1/m}|f(x)-P_n(x)|$ may be obtained in terms of n and the minimized integral. The proof is an adaptation of a method of Jackson (Transactions of this Society, vol. 22 (1921), pp. 158–166). The upper bound obtained applies uniformly to the whole infinite range of values of x. (Received March 3, 1938.)

195. J. M. Feld: *A continuous group of contact transformations containing the generalized pedal transformation.*

Lie constructed for the plane a one-parameter continuous group of contact transformations among which were included the pedal transformation and its iterates. The analytic form in which the group was expressed does not lend itself to generalization. In this paper, generalization in two directions is achieved by the use of contravariant coordinates to represent line elements. By these means, a three-parameter continuous group is constructed containing generalized skew pedal transformations and their iterates. Also a four-parameter mixed group that contains the former group as a subgroup is constructed. From the form taken by the equations defining the transformations, generalization to n-space becomes self-evident. (Received March 11, 1938.)

196. A. L. Foster: *A new representation of real numbers.*

In this paper is given a representation of real numbers in terms of a certain *characteristic* operation (see below) previously used by the author in other connections. Each finite application of the characteristic operation leads (uniquely) to a rational number, and, conversely, each rational number is so obtainable; while each infinite application leads (uniquely) to an irrational number, and, conversely. The formal properties of this characteristic decomposition of reals are in part analogous to those involved in Boolean ideal theory. Certain basic theorems of analysis, for example, the Heine-Borel theorem, are profitably reconsidered. The characteristic operation is based on the characteristic function $\psi(w, w')$ defined for all rational w, w', $(w' \neq w)$, to be the rational number $\mathfrak{a}=a_1/a_2$ interior to $[w, w']$ for which $|a_1|+|a_2|$ is minimum. (Received March 10, 1938.)

197. Philip Franklin: *Maximal real determinants.*

The maximum value of a determinant whose elements are numerically at most unity is given by Hadamard's theorem. This value is attained if the elements are complex, and also, in general, when the determinant is of order $4k$, even if the elements are restricted to be real. For values of n not divisible by four, and real elements, little about the maximum value is known beyond a guess made by Sharpe. The author here finds the maximum for the first eight orders by systematically constructing all the maximal determinants for these cases. Incidentally, the results show that the value conjectured by Sharpe is too small for $n=6$ and $n=7$. (Received March 7, 1938.)

198. Bernard Friedman: *A note on convex functions.*

It is well known that a convex function of one variable is necessarily absolutely continuous, and therefore it is immediately obvious that a convex function of many variables is absolutely continuous in the sense of Tonelli, that is, absolutely continuous in each variable separately. Whether or not it is also absolutely continuous in the sense of Carathéodory (*Vorlesungen über reele Funktionen*, p. 653) is a more difficult problem. Absolute continuity is defined by Carathéodory as follows: Let $F(x, y)$ be a measurable function and δ a square with vertices (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), and let $F(x_1, y_1)-F(x_1, y_2)-F(x_2, y_1)+F(x_2, y_2)=\Phi(\delta)$. Then $F(x, y)$ is absolutely continuous if $\Phi(\delta)$ is an absolutely continuous set function, and if $F(x, 0)$ and $F(0, y)$ are absolutely continuous. In this note a Gegenbeispiel is constructed using the integral of a continuous monotonic function defined on Cantor's middle-third set. Therefore a convex function of many variables is not necessarily absolutely continuous in the sense of Carathéodory. (Received March 17, 1938.)

199. H. L. Garabedian and W. C. Randels: *On summability by Riesz means.*

Transformations of the type $\sigma_m=(P_m)^{-1}\sum_1^m p_n s_n$, where $p_n>0$, $P_m=\sum_1^m p_n$, are considered. The problem of inclusion of such methods of summability is treated, and necessary and sufficient conditions are developed. It is shown, for instance, that if $p_n=n^\alpha$, the method is equivalent to $(C, 1)$, and if $p_n=e^n$, the method is equivalent to convergence. Necessary and sufficient conditions that summability imply convergence, and other results of a similar nature, are given. (Received March 7, 1938.)

200. H. H. Goldstine: *A generalization of the problem of Bolza in the calculus of variations.*

The problem discussed in this paper is a generalization of the familiar problem of Bolza in that the usual differential side conditions are replaced by integro-differential conditions. Analogs of the multiplier rule and its corollaries, the Euler-Lagrange equations, the Weierstrass-Erdmann corner condition, and Hilbert's differentiability condition, as well as of the necessary conditions of Weierstrass and Legendre, are obtained. The proofs are similar to the ones ordinarily given for the problem of Bolza, existence theorems for integro-differential systems replacing the theorems customarily used in the calculus of variations. (Received March 4, 1938.)

201. Saul Gorn: *Ideals in partially ordered sets, ideal extension theorems, and the projectivization of incidence geometries.* Preliminary report.

Beginning with general partially ordered sets, σ-ideals and, dually, π-ideals are defined. Each of the two types contains a subset, the principal ideals, isomorphic with the original partially ordered set. Examples: 1. The upper and lower segments of Dedekind cuts are π-ideals and σ-ideals, respectively. 2. Stone's ideals in a distributive lattice (Journal Tchécoslovaque de Mathématique et Physique, vol. 67, pp. 1–26) are σ-ideals and π-ideals. 3. The ideals in algebraic rings (with suitable partial ordering) are π-ideals. A brief calculus of ideals is developed. The following concepts are defined: ideal extension method, ideal extension; σ-lattice, σ-quasi-lattice, and their duals; complete π-ideals, regular π-ideals; and incidence geometries. Descriptive, affine, and projective geometries are examples of the last. It is shown that the "complete envelope" of MacNeille (Transactions of this Society, vol. 42 (1937), pp. 416–460) is an ideal extension, and that "projectivization" can also be performed by an ideal extension method. The regular π-ideals in an incidence geometry of dimension greater than or equal to 3 (Desargues' theorem is required for dimension 3) form a π-quasi-lattice; the σ-ideals in the result form the "projective envelope" of the incidence geometry. Additional similar results and certain lemmas, connecting modularity and dimension in lattices and Archimedean lattices and giving the dimenson of an element in an extension, are obtained. (Received March 24, 1938.)

202. J. W. Green: *Harmonic functions in domains with multiple boundary points.*

This paper is a continuation of that described in abstract 44-1-83. The method of representation of harmonic functions described therein is extended to all positive harmonic functions in a wide class of domains by using methods analogous to those of Maria and Martin (see Duke Mathematical Journal, vol. 2 (1936)). A chief result is that the representation is extended to domains whose boundary points may admit different methods of approach. (Received March 14, 1938.)

203. Mary B. Haberzetle: *Two new universal Waring theorems.*

In this paper two problems, modifications of Waring's, are considered. For the first, the summands are permitted to be not only integral nth powers, positive or zero, but also any integral powers, positive or zero, higher than the nth. For $n \geqq 9$ the number of such summands sufficient to represent every positive integer is evaluated and found to be considerably less than that required in the original Waring problem. In the second problem considered here, the summands are 0, $1+bx^n$, or their products by a, where a and b are fixed positive integers. For $19 \leqq n \leqq 400$, $1 \leqq a \leqq 4n$, $1 \leqq b \leqq 2n+1$, the number of summands sufficient to represent every positive integer is obtained. (Received March 8, 1938.)

204. J. W. Hahn and L. R. Ford: *The isotenic circle.*

Consider the way in which lengths in the four-dimensional real space of two complex variables, z_1, z_2, are altered by a non-affine projective transformation $w_1 = (a_1z_1 + a_2z_2 + a_3)/(c_1z_1 + c_2z_2 + c_3)$, $w_2 = (b_1z_1 + b_2z_2 + b_3)/(c_1z_1 + c_2z_2 + c_3)$. It is proved by the authors that each such transformation possesses a unique circle with the following property: All infinitesimal lengths at points of the circle are multiplied by a constant stretching factor λ. It follows that all angles with vertices on the circle are preserved in magnitude. There is no other point at which lengths in all directions are equally stretched. (Received March 2, 1938.)

205. D. W. Hall: *On a decomposition of true cyclic elements.*

This paper studies the decomposition of a cyclicly connected continuous curve C into the sum of certain of its closed subsets, which have been defined by G. E. Schweigert and the author and named secondary elements. Analogies and differences between the properties of these elements and the cyclic elements of G. T. Whyburn are pointed out. Among the several types of secondary elements considered the most important are the following: (a) non-degenerate true secondary elements which contain at least one non-degenerate component, and (b) degenerate true secondary elements which are infinite and totally disconnected. The following are typical theorems taken from the paper. I. C contains at most a countable number of true secondary elements. II. If E is a true secondary element of C, and K is any component of $C-E$, then K has exactly two limit points in E. III. For every preassigned positive number d and any true secondary element E, there exist in E at most a finite number of components of diameter greater than d. IV. If E be a degenerate true secondary element, then every two points of E are vertices of a theta curve in C. V. If E be a degenerate true secondary element, then $C-E$ contains an infinite sequence of components whose closures are disjoint by pairs. (Received March 9, 1938.)

206. Marshall Hall: *Equidistribution of residues in sequences.*

Assume that (u_n) is a linear, recurring sequence of order k and that the characteristic is irreducible modulo p. If $u_{n+1}, \cdots, u_{n+r}$ is a cycle of (u_n) modulo p, the residue i (mod p) occurs in this cycle $r/p+c_i$ times, where c_i is less than $p^{k/2-1}$. The result is a consequence of certain diophantine equations herein established which are comparable to those in the paper, *An isomorphism between linear recurring sequences and algebraic rings*, by the author. (Received March 3, 1938.)

207. N. A. Hall: *The solution of the trinomial equation in infinite series by the method of iteration.*

The solution of the trinomial equation as an infinite series in the coefficient has long been known as obtained by differential resolvents or by the Lagrange expansion theorem. In this paper these results are verified by a direct application of the familiar method of iteration used in numerical approximation. The kth iteration is expressed as a k-fold infinite summation by application of the binomial theorem. By rearranging this multiple sum to expand in powers of the coefficient, the coefficients of the first k powers can be expressed in a closed form by solving a relatively simple difference equation. The limit of the sum of these first k terms is shown to satisfy the original equation identically and to be of the form previously known. (Received February 24, 1938.)

208. P. R. Halmos: *Invariants of certain stochastic transformations: the mathematical theory of gambling systems.*

The "Regellosigkeit" principle of von Mises has been shown to correspond in the mathematical theory of probability to the fact that certain transformations of infinite dimensional Cartesian space into itself are measure preserving. The present paper examines properties of such transformations on more general spaces and obtains results such as the following: If a sequence of chance variables, $x_1, x_2, \cdots$, is such that the conditional expectation of x_n with respect to $x_1, \cdots, x_{n-1}$ is identically zero, then this property remains invariant under any gambling system. Applications are made to the existence and independence of "Kollektivs." A sequence of chance variables is called "asymptotically independent" if the conditional probability distributions converge to a fixed distribution. Asymptotic independence is discussed and is shown to be invariant under all transformations considered. (Received March 15, 1938.)

209. H. J. Hamilton: *Change of dimension in sequence transformations.*

The paper by the author, *Transformations of multiple sequences* (Duke Mathematical Journal, vol. 2 (1936), pp. 29–60), is referred to as H_1. It is remarked that the validity of the proofs therein developed is not affected by removal of the assumption, tacit in H_1, that the dimension of the transformed sequence is the same as that of the original sequence. There are listed: (1) certain alterations in the formal body of H_1 necessary for the wider interpretation, and (2), as an application, conditions necessary and sufficient for the transformation from (A) boundedly convergent double sequences, and (B) convergent double sequences to convergent simple sequences. (Received March 4, 1938.)

210. H. J. Hamilton: *A generalization of multiple sequence transformations.*

The results of preceding work by the author on linear transformations of n-tuple sequences of various classes into l-tuple sequences of various classes, n, l arbitrary positive integers (see abstract 44-5-209), are generalized to transformations of n-tuple sequences of the various classes into functions of l variables $x^{(\nu)}$, $(\nu=1, 2, \cdots, l)$, of analogous classes, where the range of $x^{(\nu)}$ is an infinite aggregate $E^{(\nu)}$ of any sort. Necessary and sufficient conditions for the several transformations are tabulated, and a few specializations are given. Of particular interest are the cases in which, for each ν, $E^{(\nu)}$ is the aggregate of non-negative real numbers; and in which, for each ν, $E^{(\nu)}$ is the aggregate of positive integers. In the latter case, the results reduce to those previously established. (Received March 4, 1938.)

211. O. H. Hamilton: *Concerning collections of continua in a separable space which do not cross.*

It is shown that if G is a collection of continua, all of a certain type, in a separable space, and if no two of the continua of G cross, then there is a countable subcollection H of G such that every point of S common to two continua of G is in some continuum of H. In particular the theorem holds for a collection of open curves in the plane. (Received March 15, 1938.)

212. Olaf Helmer: *Integral functions with coefficients in a given field.* Preliminary report.

There are no general algebraic relations between the zeros of an integral function $f(z)=\sum a_n z^n$ and its coefficients a_n. For, if K is any imaginary number field, and $K\langle z\rangle$ the set of all integral functions with coefficients in K, then $K\langle z\rangle$ contains a function $g(z)$ having the same zeros as $f(z)$. (If $f(z)$ has real coefficients, K may be chosen real.) $K\langle z\rangle$ is an integral domain in which some elements have infinitely many divisors. Nevertheless, the unique-factorization theorem holds: Every function is representable as a product of irreducible functions, the representation being unique apart from unit factors $e^{h(z)}$ of $K\langle z\rangle$. However, the irreducible functions of $K\langle z\rangle$ have at most two zeros (in fact, exactly one, if K is imaginary). Hence we have the unsatisfactory situation that a polynomial which is irreducible in $K[z]$ will in general be reducible in $K\langle z\rangle$. This no longer happens if we restrict ourselves to the integral subdomain $K^*\langle z\rangle$ of those functions of $K\langle z\rangle$ which are of finite order. Here, every irreducible polynomial of $K[z]$ remains irreducible in $K^*\langle z\rangle$, a fact which suggests the existence of algebraic relations between the zeros and the coefficients of integral functions of *finite* order; they will reveal themselves on studying the divisibility structure of the domains $K^*\langle z\rangle$. (Received March 11, 1938.)

213. M. R. Hestenes: *A theory of critical points of functions in the calculus of variations.*

The present paper is concerned with a theory of critical and minimax points of functions on a metric space and applications of this theory to the calculus of variations. A preliminary report of a part of this paper has been given (see abstract 42-5-186).

Since then numerous modifications and extensions have been made. Among other things, a homotopic definition of critical points is given that is analogous, but not equivalent, to that given by Morse. It is shown that every minimax point is a homotopic critical point. For functions defined on a region in a euclidean n-space these critical points are ordinary critical points. An analog of a non-degenerate critical point of index k is given for functions that are continuous but do not necessarily have derivatives. (Received March 11, 1938.)

214. Einar Hille: *Notes on linear transformations. II. Analyticity of semi-groups.*

Let E be a Banach space and $\{T_\alpha\}$ a family of bounded linear transformations on E to E forming a one-parameter semi-group, $T_\alpha T_\beta = T_{\alpha+\beta}$ for α and $\beta > 0$. Necessary and sufficient conditions that $\{T_\alpha\}$ shall be a subset of $\{W_\alpha\}$, where W_α is analytic for $\Re(\alpha) > 0$ and $\|W_\alpha\| \leqq A \exp [C|\alpha|]$, are: T_α is strongly continuous on the right for all α, and for $0 < h < \omega$ the resolvent of T_h exists outside of a circle $|\lambda - r_h| = r_h$ and is of finite order on this circle. Any transformation T whose resolvent exists outside of a circle $|\lambda - a| = |a|$ and is of finite order on the circle can be embedded in a semi-group, analytic and bounded in some right half-plane. (Received April 14, 1938.)

215. P. G. Hoel: *The χ^2 distribution for small samples.*

An expanded form of the multinomial distribution in powers of $1/N^{1/2}$ is expressed by means of its expanded generating function. From this result an expanded form of the generating function, and hence of the distribution function, of χ^2 is obtained. The terms up to, but not including, $1/N^2$ are found explicitly. The correction term thus obtained is expressed very simply by means of the usual χ^2 distribution. (Received March 11, 1938.)

216. T. R. Hollcroft: *Plane curve systems associated with singular surfaces.*

The irregularity of a system of plane curves is $I = d - d_0$, where d is the effective, and d_0 the virtual dimension of the system. The system of curves C cut from the tangent cones of a non-singular algebraic surface by a plane π has been proved irregular and the value of I found by B. Segre. In the present paper, the system C is investigated when the surface is singular. A definite value of I is obtained when the surface has distinct multiple points. Usually, only limiting values of I can be found when the surface has multiple curves. The dimensions of plane sections of a singular surface are also studied. (Received March 15, 1938.)

217. Ralph Hull: *On the units of indefinite quaternion algebras.*

The unit group G of a maximal order of an indefinite rational quaternion algebra is studied by means of an hermitian form which is the norm of the general element of the algebra. Eichler (Mathematische Annalen, vol. 114 (1936), pp. 635–654) has proved the existence for G of generators satisfying certain relations, but his methods do not yield the generators explicitly. In the present method, as in Eichler's, the group is represented as a Fuchsian group, but now the associated form makes possible the explicit determination, in any given case, of a fundamental domain for G and the corresponding generators. In particular, when G contains elements not equal to one of finite order, more complete information is obtained since the application of the method does not require the consideration of a subgroup of G free of such elements. (Received March 9, 1938.)

218. M. H. Ingraham: *On the matrix equation $TA = BT$.*

Let A and B be square matrices over a field. A treatment, believed to be new, is given for the matrix equation $TA = BT$. In particular, it is shown that the maximum rank of T is the sum of the degrees of the greatest common divisors of corresponding invariant factors of $A - \lambda I$ and $B - \lambda I$. The treatment is rational and yields as an immediate consequence the theory of similarity of matrices. The case where the elements of A and B belong to a division algebra is being studied. (Received March 10, 1938.)

219. S. B. Jackson: *The four-vertex theorem on the sphere.*

This paper extends the four-vertex theorem to spherical curves. A point on a spherical curve of class C'' is called a geodesic vertex if the geodesic curvature at the point is a relative extremum. By stereographic projection on the plane, it is shown that every simple, closed, spherical curve of class C''' contains at least four geodesic vertices. Certain spherical arcs, analogous to the plane arcs of type Ω introduced by Graustein, are given special consideration. The fundamental property established concerning them is that interior to such an arc lies at least one non-negative minimum of geodesic curvature. For spherical curves which are tangent indicatrices of simple spherical curves, the four-vertex theorem is strengthened by showing that on such a curve lie at least four geodesic transitions; that is, points or arcs for which the geodesic curvature passes through its average value. Finally, the four-vertex theorem is shown to be false for surfaces of variable Gaussian curvature by proving that on such surfaces the geodesic circles have only two geodesic vertices. (Received March 11, 1938.)

220. Nathan Jacobson: *Normal semi-linear transformations.*

Let $\mathfrak{R}$ be a vector space over a quasi-field $\mathfrak{F}$, where $\mathfrak{F}$ has an involutorial anti-automorphism $\alpha \rightarrow \bar{\alpha}$, and suppose $f = (x, y)$ is an hermitian or skew-hermitian bilinear form which is totally regular in the sense that $(u, u) = 0$ only if the vector $u = 0$. The usual theory of orthogonality holds. If T is a semi-linear transformation (s. l. t.) with automorphism S^{-1}, the adjoint of T is defined as the transformation T^* such that $(x, yT)^S = (xT^*, y)$ for all x and y. T^* is an s. l. t. with automorphism $\bar{S}$ where $\alpha^{\bar{S}} = \bar{\alpha}^{\bar{S}}$. T is called normal when TT^* is linear and equals T^*T. It is shown that such transformations are always completely reducible and in special cases are orthogonally completely reducible in the sense that the orthogonal complement of any invariant subspace is also invariant. These results have interpretations in projective geometry and in the theory of matrices. (Received March 18, 1938.)

221. Nathan Jacobson: *Simple Lie algebras over a field of characteristic zero.*

The present paper gives an extension of the theory of simple Lie algebras, over a field Φ of characteristic zero, to non-normal algebras. The problem of enumerating these algebras and determining their automorphism is discussed. Reduction is made to questions in associative algebras and these are completely solved for real closed fields and partially solved for p adic fields Φ. This involves a derivation of necessary and sufficient conditions fo ogredience of hermitian and skew-hermitian matrices with elements in a quaternion algebra over a real closed or a p-adic field. (Received March 18, 1938.)

222. Glenn James: *An increase in the lower bound of possible solutions of the Fermat equation.*

In a paper which was presented to this Society last autumn, it was proved that the first case of Fermat's last theorem is true for values of the variables less than $n(111/77)(2cn^2+1)^n$. The present paper more than doubles that limit by extending it to $n2^nc^n(111/77)(n^2+n^{1+1/n}+n^{2/n})^n$. (Received March 11, 1938.)

223. R. D. James: *The distribution of integers represented by quadratic forms.*

In this paper the following result is proved: Let d be any integer less than or equal to -3. Let $B(\xi)$ denote the number of integers $m \leqq \xi$ which are prime to d and which are represented by some binary quadratic form of discriminant d. Then there exists a positive constant b such that $B(\xi)=b\xi/(\log \xi)^{1/2}+O(\xi/\log \xi)$. The method of proof is similar to that given by Landau (Archiv der Mathematik und Physik, vol. 13 (1908), pp. 305–312) for the particular form x^2+y^2. (Received March 9, 1938.)

224. Fritz John: *A mean value theorem for second order linear differential equations with constant coefficients.*

The following theorem was proved recently by L. Asgeirsson (Mathematische Annalen, vol. 113 (1935)): For every differential equation $\sum a_{ik}u_{x_ix_k}(x_1, \cdots, x_n)=0$, with constant a_{ik}, there are pairs of quadratic surfaces of dimension p and $n-p$ respectively (where p is the index of the quadratic form $Q=\sum a_{ik}x_ix_k$), such that the mean values of any solution u, over the ellipsoids of a pair, are equal (if the variables of integration are chosen suitably). In the present paper a more general theorem is proved, which is valid irrespective of the index p of Q: The mean value of any solution u is the same for all n-dimensional ellipsoids which are confocal with respect to the "absolute" in the plane at infinity given by $Q=0$. (Received March 8, 1938.)

225. B. W. Jones: *Related genera of positive ternary quadratic forms.*

If $g=ax^2+by^2+cz^2$, where a, b, c are positive integers; if $b=Kb'$, $c=Kc'$, and $K=r^2s$, where s is without a square factor; and if $f=sax^2+b'y^2+c'z^2$; this paper shows that, if one of w simple conditions on f is satisfied, the number of classes in the genus of g is not more than w times the number of classes in the genus of f. In a large number of cases $w=1$. This can be modified to apply to forms with cross products. This result is of considerable assistance in proving the regularity of many forms g. (Received March 15, 1938.)

226. I. N. Kagno: *Perfect subdivision of surfaces.*

A graph is said to subdivide a surface perfectly if it separates the surface into 2-cells whose boundaries are circuits of the graph having n sides each, and having the property that the boundaries of any two 2-cells meet in at most a connected set. In the special case $n=3$, this subdivision is a triangulation. Necessary conditions for perfect subdivision are derived. A perfect polyhedron is defined as the 2-complex resulting from the perfect subdivision of a surface. Necessary conditions for the existence of such polyhedra are derived. In particular, the surfaces of the five Platonic solids are shown to be the only perfect polyhedra arising from the perfect subdivision of a sphere, and a non-metrical proof is given of the theorem of euclidean geometry which states that there are only five regular solids. The fundamental domain of the universal covering surface of an orientable surface is defined, and a perfect subdivision of these covering surfaces is derived. The results are applied to show that the fundamental domain of a doubly periodic function of a complex variable is either a hexagon or a quadrilateral. The concept of the 1-star of a graph is introduced, and the 1-stars of graphs which triangulate surfaces are studied. (Received March 12, 1938.)

227. M. L. Kales: *A Tauberian theorem related to Leroy summability.*

The following theorem is proved: Let the function $L(x)$ possess first and second derivatives satisfying the relations: (i) $L'(x) = o\{x^{-1}L(x)\}$, $(x\to\infty)$; (ii) $L''(x) = o\{\min(x^{-2}L(x), x^{\alpha-3}L^2(x))\}$, $(x\to\infty;\alpha>0)$. Let the sequence $u_n(t)=\exp[(nt)^\alpha L(nt) - n^\alpha L(n)]$ be non-increasing for t in the interval $0<t<1$. If $\lim_{t=1}\sum_{n=0}^\infty a_n u_n(t)=A$ and $\lim(A_m-A_n)\geqq 0$ whenever $m/n\to 1$, $(A_n=\sum_{\nu=0}^n a_\nu)$, then $\lim_{n=\infty} A_n=A$. In particular the conditions of this theorem are satisfied if $u_n(t)\equiv\Gamma(nt+1)/\Gamma(n+1)$. This is the case of Leroy summability. (Received March 8, 1938.)

228. E. R. van Kampen and Aurel Wintner: *The canonical reduction of Hamiltonian systems.*

The paper is concerned with the canonical reduction of Hamiltonian systems by means of invariant relations. (Received March 17, 1938.)

229. R. B. Kershner: *Note on ergodic curves.*

If an ϵ-ergodic curve is defined, after M. Martin, as the shortest curve passing within a distance ϵ of every point of a given plane point set M, let $L(\epsilon)$ denote its length. Then it is shown that $2\epsilon L(\epsilon)$ approaches the measure of the closure of the given point set as ϵ tends to zero. (Received March 11, 1938.)

230. M. S. Knebelman: *Contact transformations.*

A contact transformation as defined by Lie is one perserving contact of the first order, but not necessarily of zero order, between any two hypersurfaces. It is shown in this paper that generalizations of this idea as to order of contact or dimension of subvarieties do not yield anything new. A transformation preserving contact of order n between any two subvarieties is essentially the $(m-1)$th extension of Lie's transformation if the subvarieties are hypersurfaces, and it is essentially the mth extension of a point transformation if the subvarieties are of a lower dimension. (Received March 11, 1938.)

231. Hans Lewy: *On ternary forms with non-vanishing Hessian and on spherical harmonics.*

Let $f(x, y, z)$ be a real form of degree $n>2$. Assume that there exists a point $(x, y, z) \neq (0, 0, 0)$ for which $f=0$ and that the Hessian of f does not vanish anywhere except at $(0, 0, 0)$. Then there is a line l of the projective xyz-plane which does not intersect with $f=0$; and, with l as line at infinity, $f=0$ consists precisely of *one* oval. This geometrical theorem implies that the Hessian of a spherical harmonic of degree $n>2$ must vanish at some point $(x, y, z)\neq(0, 0, 0)$. (Received March 14, 1938.)

232. R. R. McDaniel: *Approximation to algebraic numbers by means of periodic sequences of transformations on quadratic forms.*

A quadratic form g of the n ordered variables $x_1, x_2, \cdots, x_n$ is defined as a normal form of the quadratic form G of n variables if the following conditions on g are satisfied: (1) the transformation that takes G into g shall be linear and of determinant $+1$; (2) $a_1 \leqq a_2 \leqq \cdots \leqq a_n$, ($a_1, a_2, \cdots, a_n$ being the coefficients of $x_1^2, x_2^2, \cdots, x_n^2$, respectively); and (3) $a_1 a_2 \cdots a_n \leqq (4/3)^e |D|$, where D is the determinant of G and $e = n(n-1)/2$. By means of a quadratic form f, whose coefficients are certain functions of an algebraic integer α and its conjugates, sequences of normal forms are developed. Each sequence has the property that there are only a finite number of different transformations changing f into successive normal forms. These normalizing transformations are closely connected with the units of the field $R(\alpha)$. The results of Minkowski's work, *On approximation to algebraic numbers by periodic transformations*, are generalized to include algebraic fields of any degree. A new proof of a form of Dirichlet's theorem on units is also given by means of the units found to be associated with certain of the transformations. (Received March 11, 1938.)

233. E. J. McShane: *Some existence theorems in the calculus of variations. II. Isoperimetric problems.*

In a previous communication (abstract 44-1-86), a method was mentioned which led to existence theorems for variation problems which did not involve the assumption of quasi-regularity. Here the same method is extended to isoperimetric problems. The problem of minimizing $\int f(x, y, \dot{y})dx$ subject to conditions $\int g^i(x, y, \dot{y})\, dx = \gamma^i$, $(i = 1, \cdots, m)$, is discussed. Subject to conditions on the order of growth of f with $|y|$, a theorem is obtained in which m is arbitrary, but all the g^i are functions of y alone. If f is regular and $m = 1$, a theorem can be established which does not involve this restriction on $g(x, y, \dot{y})$. This theorem has an analog for parametric problems which can be so generalized as to come close to covering all previously published existence theorems for isoperimetric problems in parametric form. (Received March 21, 1938.)

234. Saunders MacLane: *Axioms for lattices connected with dependence relations.*

The "exchange" lattices which arise in the abstract treatment of algebraic dependence are here further analyzed. The exchange axiom can be replaced by the requirement that $bc < a < c < b+c$ implies the existence of a quantity b_1 in the lattice such that $bc < b_1 \leqq b$ and $(a+b_1)c = a$. This form of the axiom involves no reference to the "points" of the lattice. It states in effect that the given chain of length through joining bc to $b+c$ can be subjected to two transpositions $c \rightarrow a+b_1$ and $a \rightarrow b_1$ such that the resulting chain has a second member $b_1 \leqq b$. This "transposition" form of the axiom can be extended to chains of any length and to transpositions $(a < b < c) \rightarrow (a < b' < c)$ for which $bb' = a$. A typical example of an exchange lattice is the lattice of all fields M containing a given field K and relatively algebraically closed in a fixed superfield L. If the transcendence degree of L over K is three or more, this lattice is not modular. This is true because $K(x, y)$ and $K(z, x+yz)$ intersect in K. (Received March 4, 1938.)

235. A. D. Michal: *Differentials in linear topological spaces.*

The definition of Fréchet differentials for normed linear metric spaces has been extended to a special linear topological space L only for functions with arguments and values in the same space L (cf. A. D. Michal and E. W. Paxson, Comptes Rendus (Paris), 1936; Comptes Rendus (Warsaw), 1936). The question of the differentiability of iterations of differentiable functions was not answered with complete generality. In the present paper the author gives several definitions of differentials for functions $f(x)$ on $S \subset T_1$ to T_2, where T_1 and T_2 are any two abstract linear spaces with a Hausdorff topology and with continuous operations of addition of elements and of multiplication by numbers. Many properties (including the differentiability of iterations of differentiable functions) of Fréchet differentials in Banach spaces are then shown to hold in such topological spaces. (Received March 12, 1938.)

236. A. D. Michal: *Differential geometries with linear topological coordinates.*

With the aid of the results of a former paper (cf. abstract 44-5-235) the author extends many of his theorems in general differential geometries ("Riemannian" and "non-Riemannian") with Banach coordinates to the more general corresponding cases in which the coordinate space is a linear topological space with a Hausdorff topology and with continuous operations. (Received March 12, 1938.)

237. A. D. Michal: *The differential in abstract algebra.*

Several definitions are given for the *differential* of functions with arguments and values ranging over abstract abelian groups (written additively) by making precise the phrase, *first order approximation*, in the statement, *a differential is a first order approximation to the difference.* The usual differentiability theorems for the sum of two functions and for a function of a function are shown to hold in these purely algebraic situations. (Received March 12, 1938.)

238. E. W. Miller: *A note on cubic graphs.*

Although examples have been given of a graph which contains no simple closed curve passing through all its vertices, the question of the existence of *cubic* graphs of this kind (which is of interest in connection with the four-color problem) seems not to have been settled. (See C. N. Reynolds, *Circuits upon polyhedra*, Annals of Mathematics, vol. 33 (1932), pp. 367–372.) In this note it is shown that such cubic graphs do exist. A procedure for finding such examples is developed, and a number of different sorts of examples of this general kind are obtained. (Received March 11, 1938.)

239. C. N. Moore: *An application of Hurwitz's convergence factor theorem to the generalization of Poisson's summation of Fourier series.*

In a recent note in the Comptes Rendus (vol. 205, pp. 311–313), Raphael Salem has derived a generalization of Poisson's procedure for summing Fourier series. A combination of Fejér's theorem on the summation of Fourier series and a convergence factor theorem due to Hurwitz (this Bulletin, abstract 28-4-10; cf. D. S. Morse, American Journal of Mathematics, vol. 45 (1923)) yields a theorem which is the ultimate generalization in this direction and thus includes Salem's result as a special case. (Received March 18, 1938.)

240. M. G. Moore: *Summability of expansions in solutions of differential equations.*

The study of expansions which have been called F-series in preceding papers (see abstracts 44-1-41 and 44-1-42) is continued, and a treatment is given of the Cesáro and Abel sums of the series. The results obtained include, as special cases, a number of known theorems concerning Fourier's series, as, for example, Fejér's theorem. (Received March 11, 1938.)

241. C. B. Morrey: *The differentiability of functions minimizing certain double integrals.*

In a previous paper, the author showed the existence and continuity of a vector function $z_0(x, y)=z_0^1(x, y), \cdots, z_0^n(x, y)$ which minimizes an integral $I(z, G)$ of a function $f(x, y, z^1, \cdots, z^n, p^1, \cdots, p^n, q^1, \cdots, q^n)$ over a bounded open set G among all functions z "of class D_2'" on G and subject to certain boundary conditions. It was assumed that G was bounded by a finite number of distinct Jordan curves and that f was continuous in its argument, convex in (p, q) for each fixed (x, y, z), and satisfied $m[\sum_{i=1}^n (p_i^2+q_i^2)] \leqq f \leqq M[\sum_{i=1}^n (p_i^2+q_i^2)]$. By adding various hypotheses concerning the differentiability of f and the magnitude of its second derivatives, it can be proved (1) that z_0 is of class C' on G, (2) its first derivatives satisfy uniform Hölder conditions on compact subsets of G, and (3) its second derivatives are of class L_2 on compact subsets of G. Let $z_0=z_0(x_0, y_0)$, $p_0=p_0(x_0, y_0)$, $q_0=q_0(x_0, y_0)$, and (in addition to the above) assume that the second derivatives of f satisfy Hölder conditions near $(x_0, y_0, z_0, p_0, q_0)$; then the second derivatives of z satisfy Hölder conditions near (x_0, y_0). If f is also analytic, z_0 is analytic in G. (Received March 17, 1938).

242. Marston Morse and G. A. Hedlund: *Symbolic dynamics.*

This paper is the first of a series whose aim is to extend the Poincaré-Birkhoff dynamical theories by a systematic separation of the symbolic and differential aspects. We are concerned here with the symbolic theory. The conditions on symbolic admissibility adopted cover all known cases, including Nielsen's symbolism for the case $p>1$. The space of symbolic elements is the homomorph of the Cantor perfect nowhere dense set. This is independent of the dimension. Following general theorems on stability, recurrence, and transitivity, certain special results are obtained. The recurrency function $R(n)$ is explicitly determined for the Morse recurrent sequence, the only known determination of this type. In general $R(n)>2n$ for the non-periodic case. There exist recurrent sequences for which the limit superior of $R(n)/n$ is infinite and the limit inferior finite. Transitive symbolic rays exist whose ergodic function $f(n)$ is asymptotically the least possible. If $h(n)$ is the minimum length of a block containing all blocks of n symbols, the set of transitive rays for which the limit inferior of $f(n)/h(n)$ is 1 is residual. Under the hypothesis of uniform instability, all theorems translate into differential dynamics. (Received April 1, 1938.)

243. D. C. Murdoch: *Quasi-groups which satisfy certain generalized associative laws.*

Any element A of a quasi-group G has a right unit E_A and a left unit E_A'. Quasi-groups are considered which satisfy the generalized associative laws, $A(BC) =(AB)C^{S_A}$, $(AB)C=A^{T_C}(BC)$, where $C^{S_A}=E_A C$ and $A^{T_C}=AE_C'$. Properties of unit subquasi-groups, coset expansions, and criteria for normal subquasi-groups, Jordan-

Hölder theorems, and principal chain theorems are developed. The more restrictive law, $(AB)\ (CD)=(AC)\ (BD)$, where A, B, C, D are any four elements of G, gives rise to abelian quasi-groups, in which every subquasi-group is normal. In this case all subquasi-groups containing a given unit quasi-group form a Dedekind structure. Quasi-groups with unique right unit are also considered, and examples are constructed containing any given group as subquasi-group. (Received March 17, 1938.)

244. Tadasi Nakayama and C. J. Nesbitt: *Note on symmetric algebras.*

An algebra A is called symmetric if there exists a hyperplane in A which contains all commutator elements $ab-ba$, but does not contain any right ideal (except the null ideal). (Cf. Richard Brauer and C. J. Nesbitt, Proceedings of the National Academy of Sciences, vol. 23 (1937).) For a symmetric algebra the indecomposable constituents of the first and second regular representations corresponding to the same irreducible representation are equivalent to each other. The purpose of the present note is to show that, in order that this be the case (for all irreducible representations), it is both necessary and sufficient that there exist a hyperplane in A which contains no right ideal, but which contains all elements of the form e_iae_j, $(i\neq j)$, where $e_1, e_2, \cdots, e_l$ are mutually orthogonal idempotent elements in A with the sum 1 such that the right ideals e_iA are directly indecomposable. (Observe that $e_iae_j=e_i(ae_j)-(ae_j)e_i$ is a commutator element.) (Received March 15, 1938.)

245. D. L. Netzorg: *On the number of abelian groups of a given order.*

Let $A(n)$ denote the number of abelian groups of order n. It is here shown that $(1/4)\log 5\leqq \overline{\lim}_{n\to\infty} \log A(n)\ (\log\log n/\log n)\leqq \pi/6^{1/2}$. Furthermore $\sum_{\nu=1}^{n} A(\nu) = C_1n+C_2n^{1/2}+C_3n^{1/3}+O(n^{\alpha})$, where $\alpha=29/99$, while $C_k=\prod\zeta(\nu/k)$, $(\nu=1, \cdots, \infty;\ \nu\neq k)$. The derivation of the first two terms of this development is purely elementary. (Received March 18, 1938.)

246. Ivan Niven: *A Waring problem.*

If q denotes the greatest integer less than or equal to $(3^n+1)/(2^n+1)$, then $(2^n+1)q-1$ is a sum of $J=2^{n-1}+q-1$, but not fewer, nth powers plus unity. In general, let $g(x^n+1)$ denote the number of nth powers plus unity required to express every integer. The author evaluates $g(x^n+1)$ for $n\geqq 11$. It is shown that $g(x^n+1)=J$ for the values $11\leqq n\leqq 50$. For $n>50$, a new approach is used to show that $g(x^n-1)=J$, apart from certain exceptional cases which are elaborated. (Received March 23, 1938.)

247. Rufus Oldenburger: *Representation and equivalence of forms.*

Each form $F=a_{ij\cdots m}x_ix_j\cdots x_m$ of degree p with coefficients in a field K, where K has p or more elements, can be written as $R=\sum\lambda_i(a_{ij}x_j)^p$, where $i=1, \cdots, \sigma$, σ is finite, and the λ's and a's are in K. The representation R of F is denoted by (λ, A), where λ designates the set $\lambda_1, \cdots, \lambda_\sigma$, and A the matrix (a_{ij}) with σ rows. For given F and K, let σ_m be the smallest value of σ for which F has a representation R. If for R the number σ has the value σ_m, R is termed a minimal representation of F with respect to K. The number σ_m is termed the minimal number of F for K. Under non-singular linear transformations on the variables of F with coefficients in K, the number σ_m is invariant, a minimal representation transforms into a minimal representation, and moreover a representation (λ, A) goes into a representation (λ, A') where $A'=AX$, and X is non-singular. From these properties, necessary and sufficient conditions for the equivalence of two forms may be derived. With F is associated the multilinear form $M=a_{ij\cdots m}x_iy_j\cdots z_m$. The form M may be written as $S=\sum L_\alpha M_\alpha \cdots N_\alpha$, where $\alpha=1,\cdots,\rho$, and L's, M's, $\cdots$, N's are linear forms in the x's, y's, $\cdots$, z's respectively. Let the smallest value of ρ for which M has a representation S with coefficients in K be called the factorization rank of F for K. Consider the class (F) of all forms of degree p in n essential variables. If the minimal number of F in (F) is a minimum for (F) and K, the factorization rank of F is a minimum for (F) and K, and conversely. (Received March 27, 1938.)

248. J. C. Oxtoby and S. M. Ulam: *The existence of metrically transitive transformations.* Preliminary report.

The authors set up a general method for obtaining metrically transitive measure-preserving topological transformations (homeomorphisms) for a wide variety of euclidean manifolds. (A transformation is said to be metrically transitive if there do not exist two disjoint invariant sets both having positive measure.) In particular, there is such a transformation of the n-dimensional cube, $(n\geqq 2)$, which leaves all boundary points fixed. From this one can obtain transformations of any finite pseudo-manifold or regularly connected finite complex. (For definitions, see Alexandroff-Hopf, *Topologie.*) Among measure-preserving topological transformations of the n-dimensional cube, $(n\geqq 2)$, which leave all boundary points fixed, transformations equivalent under homeomorphism to metrically transitive ones are everywhere dense in the sense of uniform approximation together with uniform approximation of the inverses. (Received March 18, 1938.)

249. H. A. Rademacher and H. S. Zuckerman (National Research Fellow): *A new proof of two of Ramanujan's identities.*

In his first paper discussing divisibility properties of the partition function $p(n)$, Ramanujan announced two identities without complete proof. Proofs were later given by Darling and Mordell. The authors prove these identities by comparison of coefficients of an expansion of each member, obtained by their variant of the Hardy-Ramanujan method. (Received March 7, 1938.)

250. H. S. Zuckerman (National Research Fellow): *Identities analogous to Ramanujan's identities involving the partition function.*

Ramanujan's identities mentioned in the preceding abstract pertain to the moduli 5 and 7. The method not only is suitable for verifications, but also can be refined to obtain new identities. Identities for the moduli 13 and 25 are found. The latter identity shows immediately that $p(25n+24)$ is divisible by 25 and also yields the result that $p(125+99)$ is divisible by 125. The identity connected with 13 makes it clear that similar congruence properties cannot be expected for the modulus 13. (Received March 7, 1938.)

251. E. D. Rainville: *Derived periodic residue systems modulo p.*

A definition of derived periodic residue systems modulo a prime is given. Such derived systems are discussed in relation to the residue systems of the ordinary derivatives of polynomials with integral coefficients. (Received March 10, 1938.)

252. E. D. Rainville: *On the representation of periodic residue systems modulo m.*

Let m and ω be any two positive integers greater than unity. Consider the set of residue systems which are periodic modulo m with period ω. A set of ω-periodic primary residue systems is constructed in terms of which each of the entire class of ω-periodic residue systems modulo m has a unique multiplicative representation. The elements considered have, under multiplication modulo m, the closure property and are associative and commutative, but contain null-divisors. The existence and construction of the subsets which act as generators do not appear to be included in any more general theory. (Received March 10, 1938.)

253. F. W. Reed: *Segmental curves and faceted surfaces.*

Based upon periodic functions defined with respect to a square, a method is developed for writing the equation of a curve having straight segments. Extension is made to curved segments and to surface forms. Theorems relating to periodicity, operators, and so on are established. Equations of standard geometric forms (polygons, polyhedrons) are found, as well as those for the figures used in design. Fourier expansions are replaced by these simple functions. (Received March 21, 1938.)

254. J. F. Ritt: *Systems of differential equations.* I. *Theory of ideals.*

The systems treated are systems in n unknown functions of a single independent variable. The equations are assumed to be analytic in the unknowns and in the variable, and algebraic in the derivatives of the unknowns. There is developed a theory of ideals which is analogous to that of Raudenbush for algebraic differential systems. (Received March 19, 1938.)

255. J. H. Roberts and N. E. Steenrod: *Monotone transformations on 2-dimensional manifolds.*

R. L. Moore has proved that if M is a 2-sphere and G is an upper semicontinuous collection of continua filling M, then G, topologized in the usual way, is homeomorphic to a cactoid, that is, to a compact continuous curve whose non-degenerate cyclic elements are 2-spheres. The authors consider the corresponding problem where M is an arbitrary compact 2-manifold. By a *generalized cactoid* is meant a compact continuous curve among whose non-degenerate cyclic elements there may be a finite number of arbitrary compact 2-manifolds, the remaining elements being 2-spheres. By a *generalized cactoid with identifications* is meant a space C obtained from a generalized cactoid C' by identification of the points in certain subsets of C', the total number of points involved being finite. The main results are as follows: In the general case, G is homeomorphic to a generalized cactoid with identifications. Conversely, if C is any generalized cactoid with identifications, then there is a 2-manifold M and an upper semicontinuous collection of continua G filling M such that G is homeomorphic to C. Special results are obtained if the elements of G are further restricted. In particular, if the Betti groups for each element of G vanish, then G is homeomorphic to M. (Received March 5, 1938.)

256. L. B. Robinson: *On the integration of Wilczynski's equations by quadratures.*

Consider the four equations (1) $\Phi(f) \equiv -y_j \partial f/\partial y_i - y_j' \partial f/\partial y_i' + \sum_{\lambda=1}^{2} (u_{\lambda i} \partial f/\partial u_{\lambda j} - u_{j\lambda} \partial f/\partial u_{i\lambda}) = 0$, $(i, j = 1, 2)$. This system is known to be passive. Write (2) $\Phi_{12}(f) = 0$, $\Phi_{ij0}(f) = 0$, $(i \geqq j)$, where $\Phi_{ij0}(f)$ is obtained from $\Phi_{ij}(f)$ by setting $y_1 = 0$. Because (1) is passive, the integration of (1) and of (2) are equivalent problems. The integrals of $\Phi_{ij0}(f) = 0$ can be obtained by quadratures. Let G be one of these integrals. A solution of $\Phi_{12}(f) = 0$ by quadratures reduces to G as initial condition. By following the method used in the above example, it is possible to integrate most of the equations studied by Wilczynski. (Received March 11, 1938.)

257. E. H. Rothe: *Theory of the topological order in some linear topological spaces*. Preliminary report.

In continuation of former papers, the present paper gives a further development of the theory of the order in some linear spaces for a certain class of representations. One of the results is that in some cases the equality of the order is not only necessary but also sufficient for the homotopy of two representations. (Received March 10, 1938.)

258. A. C. Schaeffer and R. J. Duffin: *On some inequalities of S. Bernstein and W. Markoff on the derivatives of polynomials.*

If $f_n(x)$ is a polynomial of degree n and is bounded by 1 in the interval $(-1, 1)$, a theorem of S. Bernstein states that its first derivative is less than $n/(1-x^2)^{1/2}$ in the interval. To obtain an extension of S. Bernstein's inequality for the higher derivatives of polynomials the authors consider a class of functions R which satisfy a certain differential equation. It is shown that at each point of the interval one of the functions R has a larger pth derivative than has the polynomial $f_n(x)$. By using properties of the differential equation satisfied by R it is then shown that $|f_n{}^p(x)|^2$ is not greater than the sum of the squares of the pth derivatives of $\cos (n \cos^{-1}x)$ and $\sin (n \cos^{-1}x)$ in the interval $(-1, 1)$. From these results a simple proof is found of W. Markoff's inequality on the higher derivatives of polynomials, which states that in the interval $(-1, 1)$, $|f_n{}^p(x)| \leqq |T_n{}^p(1)|$, where $T_n(x) \equiv \cos (n \cos^{-1} x)$. (Received March 10, 1938.)

259. O. F. G. Schilling: *A generalization of local class field theory.*

The author discusses in this paper the structure of the class group of normal algebras over relatively perfect fields k whose value groups $\Gamma(k)$ have finite rank over the group of rational numbers. He shows that the properties of those algebras whose indices are relatively prime to the characteristic of k are essentially determined by the nature of $\Gamma(k)$ and the multiplicative structure of the field of residue classes κ. Moreover, he discusses the different cases of fields k to which the classical theorems of local class field theory can be extended. A number of examples are given to illustrate the complexities of the additive structure of k. (Received March 11, 1938.)

260. I. J. Schoenberg: *Metric spaces and completely monotone functions.*

This paper continues previous work on the connections between problems of isometric imbedding of metric spaces and certain classes of functions (see abstract 44-1-60). The results, although intimately connected, are partly geometrical, partly purely analytical. The chief geometric result is Theorem 1: The most general continuous function $F(t)$, $F(t) \geqq 0$, $F(0)=0$, $(0 \leqq t < \infty)$, (such that if $P_0, P_1, \cdots, P_m$ are any points of the euclidean space E_m, m arbitrary, $F(\overline{P_iP_k})$, $(i, k=0, \cdots, m)$, are the distances of $m+1$ points of E_m) is of the form (1) $F(t) = (\int_0^\infty (1-e^{-t^2u})u^{-1}d\gamma(u))^{1/2}$, where $\gamma(u)$ is non-decreasing for $u \geqq 0$, and $\int_1^\infty u^{-1}d\gamma(u)$ exists. The chief analytical results are the following two theorems. Theorem 2: A function $f(t)$ is completely monotone for $t \geqq 0$, that is, $(-1)^n f^{(n)}(t) \geqq 0$, $(n=0, 1, 2, \cdots)$, for $t>0$, $f(+0)=f(0)$, if and only if $f(t^2)$ is positive definite in Hilbert space $\mathfrak{H}$, that is, $f(t^2)$ is continuous for $t \geqq 0$, and if $P_0, \cdots, P_m$ are any points of E_m, m arbitrary, then $\det \|f(\overline{P_iP_k}^2)\|_{0,m} \geqq 0$. Theorem 3:

The most general continuous function $\phi(t)$, $(0 \leqq t < \infty)$, $\phi(t) \geqq 0$, $\phi(0)=0$, such that if $f(t)$ is completely monotone for $t \geqq 0$, $f(\phi(t))$ is also completely monotone for $t \geqq 0$, is of the form (2) $\phi(t)=\int_0^\infty(1-e^{-tu})u^{-1}d\gamma$, with $\gamma(u)$ as in Theorem 1. Bochner (Duke Mathematical Journal, vol. 3, p. 498) has previously shown that a $\phi(t)$ of the form (2) has the property of this theorem. (Received March 16, 1938.)

261. I. J. Schoenberg: *On the Peano curve of Lebesgue.*

Lebesgue's example of a Peano curve (*Leçons sur l'Intégration*, pp. 44–45) is modified by defining the functions $x=x(t)$, $y=y(t)$ at once throughout the interval $0 \leqq t \leqq 1$ by two uniformly convergent series of simple continuous functions. Thus it may be presented, immediately after a discussion of "uniform convergence," to any class in advanced calculus. The note is to appear in an early issue of this Bulletin. (Received March 16, 1938.)

262. H. M. Schwartz: *On a certain class of continued fractions with applications to Bessel functions and Lommel polynomials.*

The author considers the continued fraction $\lambda_1/(x-c_1)-\lambda_2/(x-c_2)-\cdots$, ($\lambda_\kappa$, c_κ real; $\lambda_\kappa>0$), and shows that if both series $\sum\lambda_\kappa$ and $\sum|c_\kappa|$, $(\kappa=1, \cdots, \infty)$, converge, then $\Omega_n(x)/x^n$ and $\Phi_n(x)/x^n$, ($\Omega_n(x)/\Phi_n(x)$ being the nth convergent of the continued fraction), converge uniformly over the entire complex plane except at the point $x=0$. The nature of the limit functions as well as the converses of the above statements are also discussed. Applications are made to Bessel functions and Lommel polynomials. The latter are readily shown to form an orthogonal set for positive values of the parameter, with a certain explicitly given step function as the weight function. (Received March 17, 1938.)

263. Wladimir Seidel: *An example in conformal mapping.*

An example is constructed to show that there exists a closed Jordan curve C in the complex w-plane containing the origin in its interior and a closed point set E on C with the following property: If the interior region r determined by C is mapped conformally by means of the function $z=f(w)$, $f(0)=0$, $f'(0)>0$, on $|z|<1$, the set E goes over into a set of Lebesgue measure zero on $|z|=1$; if the exterior region R determined by C is mapped on $|z|>1$ by means of the function $z=F(w)$, $F(\infty)=\infty$, $F'(\infty)>0$, the set E goes over into a set of positive Lebesgue measure on $|z|=1$. (Received March 21, 1938.)

264. J. A. Shohat: *On the distribution of the zeros of orthogonal polynomials.*

Under quite general conditions it is shown that the trigonometric polynomials are the only orthogonal polynomials whose zeros $\{x_i\}$ are equidistributed with regard to the corresponding distribution function $\psi(x)$; that is, $\psi(x_2)-\psi(x_1)=\psi(x_3)-\psi(x_2)=\cdots=\psi(x_i)-\psi(x_{i-1})$, $(i=2, 3, 4, \cdots)$. (Received March 5, 1938.)

265. Virgil Snyder and Evelyn Carroll-Rusk: *A Cremona involution in S_3 without a surface of invariant points.*

In this paper an involutorial space transformation which has no surface of invariant points is considered. It is derived by means of a pencil of quadric surfaces, a generator g of a ruled surface projective with it, and g', conjugate of g as to the associated quadric. When g describes a plane curve, the semi-symmetric plane involution of Ruffini results. The space involution can be mapped upon a pencil of planes. (Received March 17, 1938.)

266. I. S. Sokolnikoff and Elizabeth S. Sokolnikoff: *Thermal stresses in elastic plates.*

It is shown that the deflection $w_0(x, y)$ of the middle surface of an elastic plate subjected to a distribution of temperature $T(x, y, z)$ satisfies the equation $\nabla^4 w_0 = \alpha_1 \nabla^2(\partial T/\partial z)_{z=0} + \alpha_2 \nabla^2(\partial Z_z/\partial z)_{z=0} + \alpha_3(\partial^3 Z_z/\partial z^3)_{z=0}$, where the stress component Z_z is the solution of $\nabla_4 Z_z = \alpha_4(\partial^2 \nabla^2 T/\partial z^2 - \nabla^4 T)$, and the α's are constants depending on the physical properties of the material. An explicit solution for Z_z is obtained under the assumption that it is of the form $Z_z = \sum_{j=0}^{n} a_j(x, y) z_j$, where n may be arbitrarily large. The equation for deflection obtained (under more severe restrictions) by A. Nadai (*Elastische Platten*, 1925) and K. Marguerre (Zeitschrift für angewandte Mathematik und Mechanik, vol. 15 (1935)) are contained in this paper as special cases. (Received March 11, 1938.)

267. J. W. T. Suckau: *A lemma on squares.*

In connection with a paper of Banach (*Sur une classe des fonctions d'ensembles*, Fundamenta Mathematicae, vol. 6 (1924), pp. 170–188) the author was led to the problem of determining the smallest positive integer k for which the following assertion is true: If we let $s_0, s_1, \cdots, s_n$ be any system of closed squares in the xy-plane such that (1) no two of the squares have common *interior* points, (2) each one of the squares $s_1, \cdots, s_n$ has at least one boundary point in common with s_0, (3) the side length of each of the squares $s_1, \cdots, s_n$ is greater than or equal to the side length of s_0; then $n \leqq k$. An elementary discussion shows that $8 \leqq k \leqq 13$. The purpose of this paper is to prove $k=8$. (Received March, 9, 1938.)

268. L. H. Swinford: *A differential equation of R. Liouville.*

R. Liouville showed (Acta Mathematica, vol. 27 (1903), pp. 55–78) that the differential equation $y' + 3n_2 y^2 + 2y^3(n_1^2 x^3 - n_2^2 x) = 0$ has no algebraic integral and conjectured that its general integral was a new function. H. Lemke (Sitzungsberichte der Berliner Mathematischen Gesellschaft, vol. 18, pp. 26–31) integrated the equation, getting x and y as functions of Jacobian elliptic functions of a parameter. By the substitution $y = dZ/dx$ the equation becomes of a class integrated by Painlevé, giving y a rational function of Jacobian elliptic functions with the argument $e^{Z/9n_2} + C$. (Received March 14, 1938.)

269. Otto Szász: *On Fourier series with restricted coefficients.*

The author considers functions $f(x)$, periodic with period 2π, and Lebesgue integrable, with the Fourier coefficients a_n, b_n satisfying the conditions $na_n \geqq -p$, $nb_n \geqq -p$, $(n=1, 2, \cdots)$, $p \geqq 0$ being given. It is known that if, in addition, $f(x)$ is bounded, then the partial sums $s_n(x)$ of the Fourier series of $f(x)$ are uniformly bounded for all n and x. Several estimates for $s_n(x)$ have been given; they are sharpened and generalized in the present paper. We prove in particular: If $|f(x)| \leqq \mu$ in $-\pi < x < \pi$, and if $va_v \geqq -p$, $vb_v \geqq -p$ for $v=1, 2, \cdots, 2n-1$, then $|s_k(x)| < (1+8/\pi)\mu + 2p(1+\log 4)$, and also $|s_k(x)| < (1+10/\pi)\mu + 2p(4 \log 2 - 1)$, for $k=1, 2, \cdots, n$. (Received March 5, 1938.)

270. Leonard Tornheim: *Integral sets of quaternion algebras over a function field.*

Let Q be a generalized quaternion division algebra over a function field of one indeterminate over F. A normalization of the basis of Q is given, and the corresponding integral sets of Dickson are determined. They are similar; and when F has characteristic 2, there is a unique integral set. Theorems on the factorization of elements in an integral set are obtained, and there is a discussion of the cases of F finite and of F the field of all real numbers. (Received March 11, 1938.)

271. Leonard Tornheim: *Sums of nth powers in fields of prime characteristic.*

The determination of those quantities of a field F with prime characteristic p which are sums of nth powers, and the number of nth powers needed, are sought. When F is finite, at most n nth powers are needed, and when $p>n$, at most $n(n+1)$ nth powers. If $n=p^f-1$ and F has more than $n+1$ elements, all quantities of F are expressible as sums of nth powers. If F is infinite and p prime to n, the same conclusion holds. There are only a finite number of perfect fields for which not every element is a sum of nth powers, and if n is a prime there is at most one exception for each value of p. For a given n all exceptions can be found simply. (Received March 11, 1938.)

272. A. W. Tucker: *Chain-deformation retracts.* Preliminary report.

Let D be a linear operator in an abstract complex K which turns a p-chain into a $(p+1)$-chain, and let $T=1-DF-FD$, where F is the customary boundary operator (cf. Lefschetz, Duke Mathematical Journal, vol. 1 (1935), pp. 1–18). Then $FT=TF$ and $Tc\sim c$ for any cycle c. If (1) $DD=0$ and (2) $DFD=D$, then $TT=T$; hence T reduces the homology theory of K to that of the part of K invariant under T. It is appropriate, therefore, to call this invariant part of K a *chain-deformation retract.* The theory of such combinatorial retracts seems to underlie the topological theorems which treat the reduction or invariance of homology. (Received March 19, 1938.)

273. H. L. Turrittin: *Approximate distribution of the zeros of certain exponential sums.*

Asymptotic approximations for the number and location of the zeros of a function $F(z)=\sum_{j=0}^{l}P_j \exp \{Q_j(z)\}$ were given by MacColl (Transactions of this Society, vol. 36 (1934), pp. 341–360), under the assumption that the Q's are polynomials in z and the P's constants. The author derives similar approximations for a function $F(z)$ generalized to the extent that the P_j's are analytic functions behaving essentially as powers of z, $|z|$ large. The zeros fall in a finite number of half-strips extending out to infinity in the complex z-plane. Three theorems, immediate extensions of those by MacColl, are given relating to the number and distribution of the generalized function. The possibility of further breaking up the half-strips into zero-free regions and narrow curvilinear bands containing the zeros is examined and first approximations to the number of zeros within the bands are made. (Received March 11, 1938.)

274. J. V. Uspensky: *Remarks on Laplace's solution of the problem of the attraction of ellipsoids.*

The first complete solution of the problem of the attraction of ellipsoids by Laplace is very remarkable in itself but rather complicated. However, with the same leading idea retained, it is possible to present the solution of the problem in a very simple and elegant manner. (Received March 9, 1938.)

275. R. W. Wagner: *On the continuity of certain matrix functions.* Preliminary report.

The solutions of the matrix equation $p(X)=A$ have been studied by Roth, Franklin, Ingraham, and others. They have found that most of the solutions, when in canonical form, have the same block structure as A; but there may, in certain cases, be solutions with a different block structure. These are called extraordinary solutions. By considering the function inverse to $p(X)$, this note shows how certain solutions of $p(X)=B$ can be made to approach some of the extraordinary solutions of $p(X)=A$ by making B approach A in the matrix space. (Received March 11, 1938.)

276. Morgan Ward and R. P. Dilworth: *Residuated lattices.*

The authors investigate lattices over which a commutative and associative multiplication, distributive with respect to union, can be defined. With any such multiplication is associated a residuation with the properties of the residual in polynomial ideal theory. Conversely, every such residuation determines a multiplication. In abstract form the principal decomposition theorems of commutative ideal theory are obtained. No non-trivial projective geometry can be residuated, and a Boolean algebra can be residuated in only one way. A detailed summary of these results will appear shortly in the Proceedings of the National Academy of Sciences. (Received March 11, 1938.)

277. Hermann Weyl and F. J. Weyl: *Meromorphic curves.*

A meromorphic curve C in projective space S_k is defined by setting up the ratios of the $k+1$ homogeneous coordinates x_i as meromorphic functions of a complex parameter z. The line of investigation, suggested by Nevanlinna's theory of meromorphic functions (case $k=1$ of the author's theory), is a modification of his approach due to Ahlfors. A characteristic function of "order" $T(r)$ is defined in terms of the intersection of the curve C with an arbitrary plane $\alpha_0x_0+\cdots+\alpha_kx_k=0$. The compensating function which makes up for the missing point $z=\infty$ is the mean value of $\log 1/\|\alpha x\|$ on the circle $|z|=r$, the distance $\|\alpha x\|$ of the plane α and the point x being defined in terms of a unitary metric. The metric is without essential influence upon $T(r)$. The first main theorem states that $T(r)$ is independent of the plane α. The second theorem establishes relations among the orders $T_l(r)$ of the curve considered as the locus of its points, tangents, and so on, $(l=1, 2, \cdots)$, and the numbers of stationary elements; they differ again from the corresponding relations for a rational curve by a compensating term. The third and most essential main theorem deals with an estimate of the compensating terms similar to Nevanlinna's famous defect relation. (Received March 11, 1938.)

278. A. H. Wheeler: *Groups of inscribed tetrahedra.*

It is shown in this paper that, in a given regular tetrahedron, there can be inscribed groups of regular tetrahedra; that, in any face of the given tetrahedron, the vertices of any selected group of inscribed tetrahedra lie on certain equilateral hyperbolas; and that these vertices are equidistant from the center of the face considered. (Received March 5, 1938.)

279. Oscar Zariski: *Uniformization in valuation rings and resolutions of singularities of an algebraic surface.*

The following two theorems form the basis of the proof: 1. Given a zero-dimensional valuation B of a field Σ of algebraic functions of two variables over an algebraically closed ground field k of characteristic zero, there exists an isomorphic mapping of Σ upon a subfield of the field $k\{u, v\}$ of meromorphic functions of elements u, v in Σ, such that B can be extended to a valuation of $k\{u, v\}$. 2. If a subring $\mathfrak{o}=k[\omega_1, \cdots, \omega_m]$ of Σ is integrally closed in Σ, the surface F; $x_i=\omega_i$, in S_m (called normal) has a finite number of singularities at finite distance. If $\mathfrak{p}=(\omega_1, \cdots, \omega_m)$ is a singular point of F, one passes to the ring $k[\omega_1, \omega_2/\omega_1, \cdots, \omega_m/\omega_1]$ and then to its integral closure $\mathfrak{o}'$, getting a new normal surface F'. The ideal $\mathfrak{o}'\mathfrak{p}$ is 1-dimensional, and if a prime zero-dimensional divisor $\mathfrak{p}'$ of $\mathfrak{o}'\mathfrak{p}$ is a singular point of F', one passes in the same manner to a new principal order $\mathfrak{o}''$ and to a normal surface F''. Virtually, one has an infinite sequence of principal orders $\mathfrak{o}\subset\mathfrak{o}'\subset\mathfrak{o}''\subset \cdots$ of normal surfaces $F^{(i)}$ and of points $\mathfrak{p}^{(i)}$ selected as indicated. The limit of the quotient rings $\mathfrak{o}^{(i)}\mathfrak{p}^{(i)}$ is a valuation ring. Theorem 1 implies that at most a finite number of the points $\mathfrak{p}^{(i)}$ can be singular on their respective surfaces. Consequently, after a finite number of steps, the singularities of F at finite distance can be resolved. (Received March 11, 1938.)

280. Max Zorn: *On a lemma about matrices.*

Every linear family of real matrices which contains the transpose and the inverse (if existent) of its elements contains a unimodular orthogonal matrix. The matrix is obtained as any unimodular matrix in the family for which the sum of the squares of the elements is a minimum. The lemma constitutes a part of the proof for the existence of a finite basis in the group of units of a hypercomplex domain of integrity. (Received March 14, 1938.)

281. E. S. Akeley: *Generating functions associated with stochastic processes.*

Given a set of cells $i=1, \cdots, r$, the transition probability p_{ij} for the transfer of an element from cell i to cell j will be assumed proportional to certain integers N_{ij}. Consider distributions of the type $(n_1, \cdots, n_r)$, where n_i represents the number of elements in i. Generating functions have been obtained for determining the transition probability $P(n_1, \cdots, n_r, n_1', \cdots, n_r')$ for the transition from $(n_1, \cdots, n_r)$ to $(n_1', \cdots, n_r')$ in case (1) the particles have identity, corresponding to the classical statistics, and (2) the particles do not have identity, corresponding to the Bose-Fermi statistics. In the first case, $\sum P(n_1, \cdots, n_r, n_1', \cdots, n_r')x_1^{n_1}\cdots x_r^{n_r}y_1^{n_1'}\cdots y_r^{n_r'}=(\sum x_iN_{ij}y_j)^n$ or $\sum P(n_1, \cdots, n_r, n_1', \cdots, n_r')y_1^{n_1}\cdots y_r^{n_r'}=\prod_i(\sum N_{ij}y_j)^{n_i}$. In the second case, the generating functions are expressed in terms of certain homogeneous polynomials associated with the function $\prod_{ij}(1-x_iy_j)^{-N_{ij}}$. Mean values of the integral powers of n_i' have been obtained in both cases. Asymptotic values using the methods of steepest descents can also be obtained when the auxiliary condition $\sum n_i' e_i=E$ is imposed on the distributions. (Received April 25, 1938.)

282. Garrett Birkhoff: *Additive functionals on lattices.*

A functional $m(x)$ defined on a lattice L is called *additive* if and only if $m(x)+m(y)=m(x\cap y)+m(x\cup y)$; *positive* if and only if $x\geqq y$ implies $m(x)\geqq m(y)$; of *bounded variation* if and only if the sums $\sum_{i=1}^{n}|m(x_i)-m(x_{i-1})|$, $(x_0<x_1<, \cdots, <x_n)$,

are bounded. Every functional on a chain is additive, and on chains the usual definition of bounded variation coincides with the above definition. Dimension and measure functions are positive and additive. Now suppose L has a zero, and require $m(0)=0$. Then the additive functions on L are a partially ordered linear space, and those of bounded variation, a linear lattice. An additive functional is of bounded variation on bounded sets if and only if it is the difference of positive additive functionals. On a complemented lattice, $m(x)$ is additive if and only if $x \cap y=0$ implies $m(x)+m(y)=m(x \cup y)$; positive if and only if $m(x) \geqq 0$ identically; and of bounded variation if and only if it is bounded. On a linear lattice, additivity in the sense of Banach implies additivity as defined above, and an additive functional is bounded in the sense of Banach if and only if it is the difference of positive additive functionals. (Received April 22, 1938.)

283. Edward Kasner: *Polygenic functions whose associated element-to-point transformations T convert the points of the γ plane into unions of the z plane.*

In preceding papers (*A new theory of polygenic functions*, Science, vol. 66 (1927), pp. 581–582; *General theory of polygenic functions*, Proceedings of the National Academy of Sciences, vol. 13 (1928), pp. 75–82; *A complete characterization of the derivative of a polygenic function*, Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 172–177) the author has shown that the derivative of a polygenic function $w=\phi(x, y)+i\psi(x, y)$ with respect to $z=x+iy$ defines an element-to-point transformation T (*the associated element-to-point transformation T of w*) from the elements of the z-plane to the points of the derivative γ-plane. The transformation T must possess these three characteristic properties: (i) the *circle* property, (ii) the *ratio* -2:1 property, (iii) the *affine-similitude* property. In this paper, the author finds *the totality of polygenic functions such that T associates with any point of the γ-plane a union of the z-plane.* These are (a) $w=f(z)$, (b) $w=-(az+b)/[\bar{a}(\bar{a}\bar{z}+\bar{b})]+cz+d$, and (c) $w=Az+B\bar{z}+C$. These transformations induce the corresponding geometric situations in the z-plane: (a) the ∞^2 point unions (stars) or the *opulence* (*the totality of ∞^3 elements*); (b) the ∞^2 circles and the ∞^1 lines through a fixed point; (c) the ∞^1 fields defined by parallel lines. The interesting solution is the new type (b) of fractional quadratic polygenic functions. (Received April 11, 1938.)

284. Edward Kasner and J. J. De Cicco: *The derivative clock congruence of a polygenic function.*

This paper is a continuation of the paper *Note on the derivative circular congruence of a polygenic function* (Kasner, this Bulletin, vol. 34 (1928), pp. 561–565). Kasner has shown that the derivative of a polygenic function $w=\phi(x, y)+i\psi(x, y)$ with respect to $z=x+iy$ is represented by a congruence of circles or clocks. A *clock* consists of a circle together with an initial (or *phase*) point. By a *radial-phase* shall be meant the phase point together with the radius of a clock. In this paper the authors find the necessary and sufficient conditions that a congruence of clocks or circles or radial-phases represents the derivative congruence of clocks or circles or radial-phases of a polygenic function w. The following conclusions are obtained: (a) a congruence of clocks represents the derivative of either none or ∞^2 polygenic functions; (b) a congruence of circles represents the derivative of either none, ∞^2, $2\infty^2$, or ∞^3 polygenic functions; (c) a congruence of radial-phases represents the derivative of either none, ∞^2, $2\infty^2$, or ∞^∞ polygenic functions. (Received April 11, 1938.)

285. R. E. Langer: *The boundary problem of linear differential systems in the complex domain.*

The differential systems considered are $y_i'(x) = \sum \{\lambda a_{i\nu}(x) + b_{i\nu}(x)\} y_\nu(x)$, the roots $r_i(x)$ of the determinant $|a_{ij}(x) - \delta_{ij} r(x)|$ being assumed distinct and non-vanishing. The variable is not restricted to be real. For suitable regions of the complex plane, within which the coefficients are analytic, asymptotic solutions of the system are obtained for a set of sectors which cover the entire plane of the complex parameter λ. This is done without the customary hypothesis that $\arg\{r_i(x) - r_j(x)\}$ be constant. The systems obtained by adjoining boundary conditions which apply at any finite set of points in the complex plane are considered. The Green's functions are constructed and the adjoint systems are defined and studied. The notion of regularity is extended and for regular systems the expansion of "arbitrary" sets of functions, that is, vectors, is discussed. In this the usual hypothesis that $\arg \{r_i(x)\}$ be constant is dispensed with. The theory given includes as special cases that in which the boundary conditions apply at just two points, or at more than two points of an interval of the axis of reals. Especially in the latter case, this formulation has distinct advantages over that given heretofore. (Received April 25, 1938.)

286. Sam Perlis: *Maximal orders in rational cyclic algebras of composite degree.*

Consider a normal division algebra A of prime-power degree $n = \pi^e > 2$ over the field R of all rational numbers, and let $q_1, \cdots, q_s$ be the ramification spots of A. In case $e > 1$ assume that π is not one of the q_i, and in case $n = 2^e$ assume that the infinite prime spot is not one of the q_i. If the q_i-index of A is n_i, then A has a cyclic generation $A = (Z, S, \sigma)$, $(\sigma = \prod q_i^{n/n_i})$, where the cyclic field Z has conductor a prime $p \equiv 1 \pmod{n}$, discriminant p^{n-1}, and the property that each q_i is a prime ideal in Z. There are n quantities y in A such that $M = Z_0 + \tau_1^{-1} y Z_0 + \cdots + \tau_{n-1}^{-1} y^{n-1} Z_0$ is a maximal order, where the τ_j are certain rational integers whose only prime factors are the q_i and Z_0 is the set of all integers of Z. Each of these n distinct maximal orders M contains the quantity u of A which satisfies $u^n = \sigma$ and generates A over Z. For the case n an odd prime these results specialize to those obtained by Hull (Transactions of this Society, vol. 38 (1935), pp. 514–530). (Received April 4, 1938.)