ON CONTINUED FRACTIONS REPRESENTING CONSTANTS*

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1. **Introduction.** Let $\xi: x^{(1)}, x^{(2)}, x^{(3)}, \cdots$ be an infinite sequence of points $x=(x_1, x_2, x_3, \cdots, x_m)$ in a space S, and let $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x), \cdots, \phi_k(x)$ be single-valued real or complex functions over S. Then the functionally periodic continued fraction

$$1+\frac{\phi_1(x^{(1)})}{1}+\frac{\phi_2(x^{(1)})}{1}+\cdots+\frac{\phi_k(x^{(1)})}{1}+\frac{\phi_1(x^{(2)})}{1}+\cdots$$
$$+\frac{\phi_k(x^{(2)})}{1}+\frac{\phi_1(x^{(3)})}{1}+\cdots$$

is a function $f(\xi)$ of the sequence ξ. By a neighborhood of a sequence ξ: $x^{(1)}, x^{(2)}, x^{(3)}, \cdots$, we shall understand a set N_ξ of sequences subject to the following conditions: (i) ξ is in N_ξ; (ii) if $\eta: y^{(1)}, y^{(2)}, y^{(3)}, \cdots$ is in N_ξ, then $\eta_\nu: y^{(\nu+1)}, y^{(\nu+2)}, y^{(\nu+3)}, \cdots$ and $\zeta_\nu: y^{(1)}, y^{(2)}, y^{(3)}, \cdots, y^{(\nu)}, x^{(\nu+1)}, x^{(\nu+2)}, x^{(\nu+3)}, \cdots$ are in N_ξ for $\nu=1, 2, 3, \cdots$.

Let $A_n(\xi)$ and $B_n(\xi)$ be the numerator and denominator, respectively, of the nth convergent of $f(\xi)$ as computed by means of the usual recursion formulas. Put

$$L(\xi, t) = B_{k-1}(\xi)t^2 + [\phi_k(x^{(1)})B_{k-2}(\xi) - A_{k-1}(\xi)]t - \phi_k(x^{(1)})A_{k-2}(\xi).$$

Then our principal theorem is as follows:

THEOREM 1. *Let there be a sequence c: $c^{(1)}, c^{(2)}, c^{(3)}, \cdots$, and a neighborhood N_c of c, and a number r having the following properties:*

(a) *$f(\xi)$ converges uniformly over N_c,*

(b) *$f(c)=r$,*

(c) *$L(\xi, r)=0$ for every sequence ξ in N_c,*

(d) *$\phi_i(x^{(\nu)})\neq 0$, $(\nu=1, 2, 3, \cdots; i=1, 2, 3, \cdots, k)$, for every sequence ξ: $x^{(1)}, x^{(2)}, x^{(3)}, \cdots$ in N_c.*

When these conditions are fulfilled, $f(\xi)=r$ throughout N_c.

The proof of Theorem 1 is contained in §2; §3 contains a specialization and §4 an application of this theorem. In §5 continued fractions

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representing constants are obtained by means of certain transformations.*

2. **Proof of Theorem** 1. Let η: $y^{(1)}, y^{(2)}, y^{(3)}, \cdots$ be any sequence in N_c. Then η_ν: $y^{(\nu+1)}, y^{(\nu+2)}, y^{(\nu+3)}, \cdots$ is in N_c, and $f(\eta_\nu)$, $(\nu=0, 1, 2, \cdots; \eta_0=\eta)$, converges by (a); and

$$(1)\qquad \begin{aligned} f(\eta_\nu) &= \frac{A_{k-1}(\eta_\nu)f(\eta_{\nu+1}) + A_{k-2}(\eta_\nu)\phi_k(y^{(\nu+1)})}{B_{k-1}(\eta_\nu)f(\eta_{\nu+1}) + B_{k-2}(\eta_\nu)\phi_k(y^{(\nu+1)})}, \\ f(\eta_{\nu+1}) &= -\frac{B_{k-2}(\eta_\nu)f(\eta_\nu) - A_{k-2}(\eta_\nu)}{B_{k-1}(\eta_\nu)f(\eta_\nu) - A_{k-1}(\eta_\nu)}\phi_k(y^{(\nu+1)}). \end{aligned}$$

The determinant of the matrix

$$\begin{pmatrix} A_{k-1}(\eta_\nu), & A_{k-2}(\eta_\nu)\phi_k(y^{(\nu+1)}) \\ B_{k-1}(\eta_\nu), & B_{k-2}(\eta_\nu)\phi_k(y^{(\nu+1)}) \end{pmatrix}$$

is $\pm\phi_1(y^{(\nu+1)})\phi_2(y^{(\nu+1)}) \cdots \phi_k(y^{(\nu+1)})$ and is therefore $\neq 0$ by (d). Hence the denominators in (1) cannot vanish, for otherwise the numerators would also vanish, which is impossible. It then follows from (c) that if $f(\eta_\nu)=r$ for one value of ν, then $f(\eta_\nu)=r$ for all values of $\nu(=0, 1, 2, 3, \cdots)$. In particular, if ζ_ν is the sequence $y^{(1)}, y^{(2)}, y^{(3)}, \cdots, y^{(\nu)}, c^{(\nu+1)}, c^{(\nu+2)}, c^{(\nu+3)}, \cdots$, then $f(\zeta_\nu)=r$, $(\nu=1, 2, 3, \cdots)$.

Now by (a), for every $\epsilon>0$ there exists a K such that if $n>K$, $p=1, 2, 3, \cdots$,

$$(2)\qquad \left| \frac{A_{n+p}(\zeta_\nu)}{B_{n+p}(\zeta_\nu)} - \frac{A_n(\zeta_\nu)}{B_n(\zeta_\nu)} \right| < \epsilon$$

for $\nu=1, 2, 3, \cdots$. Choose a fixed $n>K$, and then choose ν so large that $A_n(\zeta_\nu)/B_n(\zeta_\nu)=A_n(\eta)/B_n(\eta)$. Then on allowing p to increase to ∞ in (2) we find that

$$\left| f(\zeta_\nu) - \frac{A_n(\eta)}{B_n(\eta)} \right| \leqq \epsilon \quad \text{or} \quad \left| r - \frac{A_n(\eta)}{B_n(\eta)} \right| \leqq \epsilon$$

if $n>K$. That is, $f(\eta)=r$. Since η was any sequence in N_c our theorem is proved.

* Leighton and Wall, *On the transformation and convergence of continued fractions*, American Journal of Mathematics, vol. 58 (1936), pp. 267–281; Wall, *Continued fractions and cross-ratio groups of Cremona transformations*, this Bulletin, vol. 40 (1934), pp. 587–592.

3. **Specialization of Theorem** 1. Let the sequence c be such that $f(c)$ is a periodic continued fraction of period k. Let r, s be the roots of the quadratic equation $L(c, t)=0$. Then* in order for $f(c)$ to converge to the value r the following two conditions are both necessary and sufficient, namely:

(α) $B_{k-1}(c)\neq 0$,

(β) $r=s$ or else
$|B_{k-1}(c)r+\phi_k(c^{(1)})B_{k-2}(c)|>|B_{k-1}(c)s+\phi_k(c^{(1)})B_{k-2}(c)|$ and $A_\lambda(c)-sB_\lambda(c)\neq 0$, $(\lambda=0, 1, 2, \cdots, k-2)$.

An important and simple sufficient condition† for the uniform convergence of $f(\xi)$ over N_c is that

(γ) $|\phi_i(x^{(\nu)})|\leq\frac{1}{4}$, $(i=1, 2, 3, \cdots, k;\ \nu=1, 2, 3, \cdots)$, for every sequence ξ: $x^{(1)}, x^{(2)}, x^{(3)}, \cdots$ in N_c.

From these remarks and Theorem 1 we then have this result:

THEOREM 2. *Let there be a sequence c and a neighborhood N_c of c such that* (γ) *and conditions* (c), (d) *of Theorem* 1 *hold. Then if $f(c)$ is a periodic continued fraction of period k, we have $f(\xi)=r$ throughout N_c.*

4. **Application in the case where $\phi_1, \phi_2, \phi_3, \cdots, \phi_k$ are polynomials.** If $k=1$, then $L(\xi, t)=t^2-t-\phi_1(x^{(1)})$, so that in order for (c) of Theorem 1 to hold ϕ_1 must be a constant, and $f(\xi)$ reduces to an ordinary periodic continued fraction.

Let $k=2$. Then $L(\xi, t)=t^2+[\phi_2(x^{(1)})-\phi_1(x^{(1)})-1]t-\phi_2(x^{(1)})$. We shall suppose that $\phi_\nu(x)=\phi_\nu(x_1, x_2, x_3, \cdots, x_m)$, $(\nu=1, 2)$, are polynomials in the real or complex variables $x_1, x_2, x_3, \cdots, x_m$. Let a, b be the constant terms, and G, H the coefficients of $x_1^u x_2^v \cdots x_m^w$ in ϕ_1 and ϕ_2, respectively. Then (c) of Theorem 1 is equivalent to the relations

$$(b-a)r-b=r(1-r), \quad (H-G)r-H=0, \qquad \text{all } G, H.$$

If $r=0$, then $\phi_2\equiv 0$, while if $r=1$, then $\phi_1\equiv 0$. Suppose $r\neq 0, 1$. Then if either G or H is 0, the other is 0 also, and if $G=H$, their common value is 0. Hence (c) of Theorem 1 takes the form of the following identity:

$$r\phi_1\equiv(r-1)(\phi_2+r), \qquad r\neq 0, 1. \tag{3}$$

On referring to Theorem 2 we now have this result:

THEOREM 3. *Let $\phi_1(x)$ and $\phi_2(x)$ be polynomials in the real or complex variables $x_1, x_2, x_3, \cdots, x_m$ connected by the identity* (3) *with con-*

* Perron, *Die Lehre von den Kettenbrüchen*, 1st edition, p. 276.

† Perron, loc. cit., p. 262.

stant terms a and b, respectively. Let r, in (3), *and s be the roots of the quadratic equation* $t^2+(b-a-1)t-b=0$ *such that* $r=s$ *or else* $|r+b| > |s+b|$, $s\neq 1$. *Let a, b be such that* $|a|<\frac{1}{4}$, $|b|<\frac{1}{4}$, $a\neq 0$, $b\neq 0$. *Then there exists a positive constant R such that throughout the circle* $|x_i^{(\nu)}|\leqq R$, $(i=1, 2, \cdots, m; \nu=1, 2, \cdots)$, *we have*

$$(4)\qquad 1+\frac{\phi_1(x^{(1)})}{1}+\frac{\phi_2(x^{(1)})}{1}+\frac{\phi_1(x^{(2)})}{1}+\frac{\phi_2(x^{(2)})}{1}+\cdots\equiv r,$$

$$x^{(\nu)}=(x_1^{(\nu)}, x_2^{(\nu)}, \cdots, x_m^{(\nu)}).$$

In applying Theorem 2 we have taken $c^{(\nu)}=(0, 0, 0, \cdots, 0)$ in the sequence c. It is to be observed that, when this is done and Theorem 2 applies, the value of the continued fraction depends upon only the constant terms of the polynomials $\phi_1, \phi_2, \phi_3, \cdots, \phi_k$.

5. **Singular continued fractions.** Let T be a transformation which carries the continued fraction $f=x_0+K(x_i/1)$ into another continued fraction $Tf=x_0'+K(x_i'/1)$ in such a way that when either f or Tf converges the other does also and their values are equal. We shall speak of such a transformation as a *proper* transformation of f. Suppose moreover that for some positive integer n the elements x_i of f are subject to the condition

$$(5)\qquad x_i=x_i',\qquad i=n, n+1, n+2, \cdots.$$

This gives the following formal relation:

$$x_0+\frac{x_1}{1}+\cdots+\frac{x_{n-1}}{g_n}=x_0'+\frac{x_1'}{1}+\cdots+\frac{x_{n-1}'}{g_n},$$

from which one may compute the value of the continued fraction

$$g_n=1+\frac{x_n}{1}+\frac{x_{n+1}}{1}+\cdots$$

when the latter converges.

The procedure outlined above will now be carried out for the following proper transformation:*

$$T_2:\quad \begin{aligned} &x_0'=x_0+x_1,\qquad x_1'=-x_1,\qquad x_2'=(1+x_3)/x_2;\\ &x_{2n+1}'=x_{2n+1},\qquad x_{2n+2}'=(1+x_{2n+1})(1+x_{2n+3})/x_{2n+2},\\ &\qquad n=1, 2, 3, \cdots;\ x_n\neq 0, -1 \text{ if } n>0.\end{aligned}$$

In this case the relations (5) are satisfied if and only if

* Leighton and Wall, loc. cit., p. 277.

$$(6)\qquad x_{2i+2}^2 = (1 + x_{2i+1})(1 + x_{2i+3}), \qquad i = n, n+1, n+2, \cdots,$$

where if $n=0$ the first of these relations is to be replaced by $x_2^2=(1+x_3)$. When $n=0$ we have the relation

$$x_0 + x_1/g_2 = x_0 + x_1 - x_1/g_2$$

from which to compute g_2. It follows that, if f converges, g_2 must converge and have the value 2; and if g_2 converges to a value different from $0, f$ must converge and $g_2=2$. Moreover, it is impossible for g_2 to have the value ∞, for that would imply that $f=x_0$ while $Tf=x_0+x_1\neq f$. If we now write out the continued fraction g_2 and make a change in notation, the following theorem results.

THEOREM 4. *If $x_1, x_2, x_3, \cdots$ are arbitrary complex numbers $\neq 0, -1$, then the continued fraction*

$$(7)\qquad \begin{aligned} &1 + \frac{e_1(1+x_1)^{1/2}}{1} + \frac{x_1}{1} + \frac{e_2[(1+x_1)(1+x_2)]^{1/2}}{1} + \frac{x_2}{1} \\ &+ \frac{e_3[(1+x_2)(1+x_3)]^{1/2}}{1} + \cdots, \qquad e_i = \pm 1, \end{aligned}$$

has one of the values 0 or 2 whenever it converges, and it cannot diverge to ∞.

It is interesting to observe that if $e_i=+1$, (7) is the formal expansion of 2 into a continued fraction by means of the identity

$$1 = \cfrac{(1+t)^{1/2}}{1 + \cfrac{t}{1+(1+t)^{1/2}}}.$$

As a special case we have the expansion

$$(1+N)^{1/2} = 1 + \frac{N}{1} + \frac{N+1}{1} + \frac{N}{1} + \frac{N+1}{1} + \cdots,$$

which is valid if N is a positive integer.

The transformation T_2 is one of an infinite group of transformations discussed by the writer* elsewhere in this Bulletin. If one obtains the singular continued fractions corresponding to the case $m=3$ (in the notation of §3, p. 589, of that article), the following three theorems result.

* Wall, loc. cit.

THEOREM 5. *If the continued fraction*

$$1-\frac{x_1}{1}-\frac{(x_1^2-x_1+1)}{1}-\frac{x_1}{1}-\frac{x_2}{1}-\frac{(x_2^2-x_2+1)}{1}-\frac{x_2}{1}-\frac{x_3}{1}-\cdots, \qquad x_n\neq 0,\ x_n^2-x_n+1\neq 0,$$

converges, its value is $(1\pm i3^{1/2})/2$.

THEOREM 6. *If the continued fraction*

$$1-\frac{e_1}{1}-\frac{x_1}{1}-\frac{(2-x_1)}{1}-\frac{e_2}{1}-\frac{x_2}{1}-\frac{(2-x_2)}{1}-\frac{e_3}{1}-\cdots,$$

$$e_n=\pm 1,\quad x_n\neq 0,\ 2,$$

converges, its value is 0 *or* 1.

THEOREM 7. *If the continued fraction*

$$1-\frac{x_1}{1}-\frac{(1-2x_1)}{1}-\frac{x_1}{1}-\frac{x_2}{1}-\frac{(1-2x_2)}{1}-\frac{x_2}{1}-\cdots,$$

$$x_n\neq 0,\ \tfrac{1}{2},$$

converges, its value is 0 *or* $\frac{1}{2}$.

The proofs of these theorems are along the lines of the proof of Theorem 4, and will be omitted.

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