PROOF OF THE NON-ISOMORPHISM OF TWO COLLINEATION GROUPS OF ORDER 5184*

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Introduction. Let S denote the collineation

$$\rho x_r = \epsilon^{r-1} x_r', \quad (r = 1, \cdots, n), \qquad \epsilon = \cos(2\pi/n) + i \sin(2\pi/n),$$

and T the collineation

$$\rho x_r = x_{r+1}', \qquad (r = 1, \cdots, n), \qquad x_{n+1}' \equiv x_1'.$$

The abelian group $\{S, T\}$ of order n^2 is invariant under a group† C_n of order

$$n^5\left(1 - \frac{1}{p_1^2}\right)\left(1 - \frac{1}{p_2^2}\right) \cdots \left(1 - \frac{1}{p_m^2}\right),$$

where $p_1, p_2, \cdots, p_m$ are the distinct prime factors of n. The order of C_6 is 5184.

Winger‡ has discussed briefly the monomial group of order $(r+1)!n^r$ that leaves invariant the variety

$$x_0^n + x_1^n + x_2^n + \cdots + x_r^n = 0.$$

This group is generated by the symmetric group of degree $r+1$ and an abelian group of order n^r in canonical form. For $r=3$ and $n=6$ there results a group G of order 5184 which has been treated by Musselman.§ The purpose of this note is to prove that G and C_6 are not simply isomorphic. The proof consists in showing that the number of collineations of period 2 in G exceeds the number of collineations of period 2 in C_6.

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† In fact, C_n is the largest collineation group in n variables containing $\{S, T\}$ invariantly, the coefficients and variables being in the field of complex numbers. (Author's dissertation, Ohio State University, 1934.)

‡ *Trinomial curves and monomial groups*, American Journal of Mathematics, vol. 52 (1930), p. 394.

§ *On an imprimitive group of order* 5184, American Journal of Mathematics, vol. 49 (1927).

Proof of the Non-Isomorphism of G and C_6. The group C_6 is generated by $\{S, T\}$ and the two collineations

$$V: \quad \rho x_r = \sum_{c=1}^{6} \epsilon^{(r-1)(c-1)} x_c' , \quad (r = 1, \cdots, 6),$$
$$W: \quad \rho x_r = \epsilon^{-(r-1)^2/2} x_r' , \quad (r = 1, \cdots, 6),$$

satisfying the following relations:

$$V^4 = W^{12} = 1,\ V^2W = WV^2,\ V^{-1}SV = T^{-1},\ W^{-1}SW = S,$$
$$(VW)^3 = V^2 = (WV)^3,\ W^6 = S^3,\ V^{-1}TV = S,\ W^{-1}TW = S^{-1}T.$$

The order of $H = \{V, W\}$ is 576. This group may be constructed by the following chain of invariant subgroups and an independent proof that the order of C_6 is 5184 follows readily.

$$H = \{V, G_{288}\},\quad G_{288} = \{W^5VW^3V^3, G_{96}\},\quad G_{96} = \{W^2, G_{32}\},$$
$$G_{32} = \{W^2(W^2V)^2, G_{16}\},\ G_{16} = \{(W^2V)^3V, G_4\},\ G_4 = \{S^3, T^3\}.$$

Since G_4 is contained in $\{S, T\}$ which is invariant under H, the order of C_6 is $576 \cdot 36/4 = 5184$.

If Q, of order 144, represents the quotient group of C_6 with respect to $\{S, T\}$, each element of Q, being a co-set of C_6, represents 36 collineations of C_6 that transform $\{S, T\}$ into itself according to the same isomorphism of $\{S, T\}$ with itself.* There are 24 collineations S^jT^k of period 6 in $\{S, T\}$; if S is transformed into a particular S^jT^k, the collineation S^lT^m into which T is to be transformed may be selected in six ways. Let K represent a class of 144 collineations of C_6 corresponding to the 144 distinct possible sets (j, k, l, m). That is, K contains one and only one collineation from each of the 144 augmented co-sets of C_6. The square of $A \cdot S^rT^s$, an arbitrary collineation of the class K from the co-set to which A belongs, may be expressed in the form $A^2S^uT^v$ and hence is of period 2 only if A^2 is in $\{S, T\}$. That is, a necessary condition that $A \cdot S^rT^s$ be of period 2 is that A^2 be commutative with both S and T. Among any class K there are only 8 collineations B such that the corresponding sets of values (j, k, l, m) satisfy the congruences arising from the conditions that B^2 transform S into S and T into T.

* It may easily be proved that the 36 collineations of $\{S, T\}$ are the only collineations in six variables commutative with both S and T.

The following table shows 8 such collineations, their squares, and the collineations of $\{S, T\}$ which multiply these 8 collineations on the right to form collineations of C_6 of period 2. The numbers in the last column show the total number of collineations of C_6 of period 2 corresponding to each B of K. Thus it is seen that C_6 contains just 99 collineations of period 2.

It is easily shown that G contains more than 99 collineations of period 2 and hence G and C_6 are not simply isomorphic.

W^3	$W^6=S^3$	T^3, S^3T^3	2
S^3	$S^6=1$	1, T^3, S^3T^3	3
$U^3=V^{-1}W^3V$	$U^6=T^3$	S^3, S^3T^3	2
$X=WVW^3VW^5 \begin{Vmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{Vmatrix}$	$X^2=1$	S^jT^k where (j, k) satisfies the congruence $3(j+k)\equiv 0 \pmod 6$	18
$V^2=R$	$V^4=1$	S^jT^k, $(j, k=1, \cdots, 6)$	36
$RX=XR$	$(RX)^2=1$	1, S^3T^3	2
$RW^3=W^3R$	$(RW^3)^2=W^6$	S^jT^k $(j=1, \cdots, 6)$ $(k=1, 3, 5)$	18
$RU^3=U^3R$	$(RU^3)^2=T^3$	S^jT^k $(j=1, 3, 5)$ $(k=1, \cdots, 6)$	18
			99

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